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ORIGINAL ARTICLE

j-filters and j-congruences of Locally Bounded $\underline{K}_2$-algebras

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Abstract

In this paper, we introduce and characterize the notions of j-filters and principal j-filters of a locally bounded $\underline{K}_2$-algebra L with $L^{\vee}=[j)$. Many properties of j-filters of a locally bounded $\underline{K}_2$ algebra L are investigated, and a set of equivalent conditions for a filter F to be a j-filter is given. Also, we show that the class $F_j(L)$ of all j-filters of L forms a bounded modular lattice. We obtain many interesting properties of the principal j-filters of a locally bounded $\underline{K}_2$-algebra L. Moreover, a characterization of a j-filter of a locally bounded $\underline{K}_2$-algebra L is given in terms of principal j-filters of L. We establish and characterize the lattice $Con_j(L)$ of all j-lattice congruences of a locally bounded $\underline{K}_2$-algebra L via j-filters and the lattice $Con_j^p(L)$ of all principal j-lattice congruences via principal j-filters of L. Finally, we prove that the principal j-lattice congruence $\theta_{(x)^{\Delta}}, x\in L$ is a $\{\circ\}$-congruence on L if and only if x is a Boolean element of L such that $x\leq j^{\circ\circ}$.

Keywords: $\underline{K}_2$-algebras, K_2-algebras, Congruences, Filters, Lattice congruences, Modular GMS-algebras, GMS-algebras, MS-algebras

1. Introduction

The class MS of all MS-algebras, which is a generalization of the class M of all de Morgan algebras and the class S of all Stone algebras, was introduced by Blyth and Varlet [1]. The subvarieties of the class MS were characterized by Blyth and Varlet in Ref. [2]. Additionally, Blyth and Varlet [3,4] constructed MS-algebras from the subclass K_2 by using quadruples. More basic properties of MS-algebras are considered in Refs. [5–7]. The class GMS of all generalized MS-algebras was investigated by Ševčovič [8]. Later, Badawy [9] introduced and constructed the principal generalized K_2-algebras (briefly principal GK_2-algebras) from generalized Kleene algebras and bounded lattices using triples. Also, Badawy [10] constructed $\underline{K}_2$-algebras from Kleene algebras and modular lattices by means of $\underline{K}_2$-quadruples. He characterized the isomorphism of $\underline{K}_2$-algebras in terms of $\underline{K}_2$-quadruples.

In [11], the author studied the d_L-filters of a principal MS-algebras, that properly each d_L-filter of L contains the dense filter $D(L)=d_L$, but in a locally bounded $\underline{K}_2$-algebra L with $L^{\vee}=[j)$ each j-filter contains $L^{\vee}$, where $D(L)\subseteq L^{\vee}$, so every j-filter is a d_L-filter but the converse is not true.

Many properties of filters are studied in p-algebras and MS-algebras are given in Refs. [12–14].

El Fawal *et al*. [15] introduced and characterized $\underline{K}_2$-congruence pairs of modular generalized MS-algebras from the class $\underline{K}_2$ of all $\underline{K}_2$-algebras.

In this paper, we introduce the concept of j-filters of a locally bounded $\underline{K}_2$-algebra L with $L^{\vee}=[j)$. We show that the set $F_j(L)$ of all j-filters of a locally bounded $\underline{K}_2$-algebra L forms a bounded modular lattice. We introduce the notion of principal j-filters of L and investigate the basic properties of such filters . Also, we prove that $F_j^p(L)$ of all principal j-filters of L forms a Kleene algebra and it is a bounded sublattice of $F_j(L)$. Moreover, we study the relationship between the relation ψ, where

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$(x,y) \in \psi \Leftrightarrow x^\circ = y^\circ$,

and the principal j-filters of L. Further, we characterize the lattice $Con_j(L)$ of all j-lattice congruences of a locally bounded $\underline{K}_2$-algebra L via j-filters and the lattice $Con_j^p(L)$ of all principal j-lattice congruenc via principal j-filters of L. Finally, we prove that the principal j-lattice congruence $\theta_{(x)^\Delta}$ is a congruence (that preserving $\vee, \wedge, \circ$) on L if and only if x is a Boolean element of L such that $x \leq j^{\circ\circ}$.

2. Preliminaries

This section contains several definitions and important results that are essential to this work and are mostly considered in Refs. [10–12].

Definition 1. [16] *A filter F is a nonempty subset of L, that satisfies the conditions:*

(1) $x, y \in F$ implies $x \wedge y \in F$,
(2) If $x \geq y$, $y \in F$ and $x \in L$, then $x \in F$

A filter $[A)$ generated by a subset A of a lattice L is defined as follows:
$[A) = \{x \in L : x \geq a_1 \wedge a_2 \wedge ... \wedge a_n$, for some $a_i \in A, i = 1, 2, ..., n\}$
If $A = \{a\}$, we write $[a)$ instead of $[\{a\})$ and $[a) = \{x \in L : x \geq a\}$ is called a principal filter generated by a.
The lattice $(F(L); \wedge, \vee)$ of all filters of a lattice L is a distributive (modular) if and only if the lattice L is distributive (modular), where

$F_1 \wedge F_2 = F_1 \cap F_2$ and
$F_1 \vee F_2 = \{x \in L : x \geq f_1 \wedge f_2$, for some $f_1 \in F_1, f_2 \in F_2\}$.

Now, we recall the definitions of MS-algebras, K_2-algebras, de Morgan algebras, and stone algebras from Ref. [17].
An MS-algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ is an algebra with type $(2,2,1,0,0)$, where $(L; \vee, \wedge, 0, 1)$ is a distributive lattice and the unary operation $^\circ$ satisfies the following:

$x \leq x^{\circ\circ}$, $(x \wedge y)^\circ = x^\circ \vee y^\circ$, $1^\circ = 0$.

An MS-algebra together with the identity $x = x^{\circ\circ}$ is a de Morgan algebra.
A Kleene algebra is a de Morgan algebra which satisfies this identity

$(x \wedge x^\circ) \vee (y \vee y^\circ) = y \vee y^\circ$.

An MS-algebra that satisfies the following two identities is a K_2-algebra:

$x \wedge x^\circ = x^\circ \wedge x^{\circ\circ}, (x \wedge x^\circ) \vee (y \vee y^\circ) = y \vee y^\circ$.

An MS-algebra is called a Stone algebra if it satisfies the identity $x \wedge x^\circ = 0$ and is called a Boolean algebra if it satisfies this identity $x \vee x^\circ = 1$.
A generalized de Morgan algebra $(L; \vee, \wedge, ^-, 0, 1)$ (or GM-algebra) is a bounded lattice $(L; \vee, \wedge, 0, 1)$ with the unary operation $^-$ satisfies the identities:

$x = \overline{\overline{x}}$, $\overline{(x \wedge y)} = \overline{x} \vee \overline{y}$ and $\overline{1} = 0$.

A generalized MS-algebra (simply GMS-algebra) is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation $^\circ$ satisfies the identities:

$x \leq x^{\circ\circ}$, $(x \wedge y)^\circ = x^\circ \vee y^\circ$ and $1^\circ = 0$.

We observe that a modular GMS-algebra L is a GMS-algebra, that is $(L; \vee, \wedge, 0, 1)$ is a modular lattice. The class GMS contains the classes GM and $\underline{S}$ of all modular S-algebras. Moreover, the class MS is a proper subclass of the class GMS of all GMS-algebras.

Theorem 2. [8] *For any two elements a, b of a GMS-algebra L, we have*

(1) $0^\circ = 1$,
(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$,
(6) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$.

An element a of *a GMS-algebra L* is called a closed element of L if $a = a^\circ$. The set $L^{\circ\circ}$ of all closed elements of L is defined by $L^{\circ\circ} = \{a \in L : a = a^{\circ\circ}\}$. It is known that $(L^{\circ\circ}; \vee, \wedge, ^\circ, 0, 1)$ is a GM-algebra. The element $d \in L$ is called a dense element of L if $d^\circ = 0$.
The class of all $\underline{K}_2$-algebras was presented by Badawy [10], as a common abstract of Kleene algebras and modular S-algebras (modular p-algebras that satisfy the stone identity $x^\circ \vee x^{\circ\circ} = 1$) as follows:

Definition 3. [10] *A $\underline{K}_2$-algebra L is a modular GMS-algebra such that $L^{\circ\circ}$ is a distributive lattice and L satisfies the following:*

$x \wedge x^\circ = x^\circ \wedge x^{\circ\circ}, x \wedge x^\circ \leq y \vee y^\circ$.

We will denote by $\underline{\mathbf{K}}_2$ for the class of all $\underline{K}_2$-algebras. It is clear that $\underline{K}_2$ contains the classes $\mathbf{K}_2$, $\mathbf{S}$, $\mathbf{M}$, $\mathbf{K}$, $\mathbf{B}$ and $\underline{\mathbf{S}}$.

Theorem 4. [10] *Let L be a $\underline{K}_2$-algebra . Then we have*

(1) $x = x^{\circ\circ} \wedge (x \vee x^\circ)$, for all $x \in L$,
(2) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ is a Kleene algebra,
(3) $L^\vee = \{x \in L : x \vee x^\circ\} = \{x \in L : x \geq x^\circ\}$ is a filter of L,

(4) $L^{\wedge} = \{x \in L : x \wedge x^{\circ}\} = \{x \in L : x \leq x^{\circ}\}$ is an ideal of L,
(5) $D(L) = \{x \in L : x^{\circ} = 0\}$ is a filter of L and $D(L) \subseteq L^{\vee}$.

Remark 5. *In a $\underline{K}_2$-algebra L, the condition $x \wedge x^{\circ} \leq y \vee y^{\circ}$, where $x \wedge x^{\circ} \in L^{\wedge}$ and $y \vee y^{\circ} \in L^{\vee}$ means that $a \leq z$ for every $a \in L^{\wedge}$ and $z \in L^{\vee}$.*

Definition 6. [18] *On a lattice L, the lattice congruence θ is an equivalence relation with the following condition:*

if $(a, b) \in \theta, (c, d) \in \theta$ imply $(a \vee c, b \vee d) \in \theta$, $(a \wedge c, b \wedge d) \in \theta$.

Theorem 7. [16] *The smallest lattice congruence $\theta_{(x,y)}$ on a lattice L that identifies x and y is called a principal lattice congruence on L and is defined by*

$(a, b) \in \theta_{(x,y)} \Leftrightarrow a \wedge x \wedge y = b \wedge x \wedge y$ and $a \vee x \vee y = b \vee x \vee y$.

It is clear that,

$(a, b) \in \theta_{(x,1)} \Leftrightarrow a \wedge x = b \wedge x$.

Definition 8. [18] *For any lattice congruence θ on a bounded lattice L,* the Cokernel of θ (briefly Cokerθ) is the set $\{x \in L : (x, 1) \in \theta\}$, which forms a filter of L.

Lemma 9. [15] *Let L be a $\underline{K}_2$-algebra with $L^{\vee} = [j)$. Then we have*

(1) $x = x^{\circ\circ} \wedge (x \vee j)$, for all $x \in L$,
(2) $(a \wedge b) \vee j = (a \vee j) \wedge (b \vee j)$, for all $a, b \in L^{\circ\circ}$,
(3) $(x \wedge y) \vee d = (x \vee j) \wedge (y \vee j)$, for all $x, y \in L$.

A lattice congruence θ is called a congruence on a $\underline{K}_2$-algebra L if $(a, b) \in \theta$ implies $(a^{\circ}, b^{\circ}) \in \theta$. For any $\underline{K}_2$-algebra L, we denote by $Con(L)$ the lattice of all congruences of L and by $Con_{lat}(L)$ the lattice of all lattice congruences of L. Additionally, we use Δ_L and ∇_L, respectively , to indicate the identity congruence $\{(a, a) : a \in L\}$ and the universal congruence $L \times L$ on L.
Throughout the paper, we consider $L^{\vee} = [j)$ to be the principal filter of a locally bounded $\underline{K}_2$-algebra L that is generated by j.
For more information for filters, congruences on lattices and MS-algebras, see the references [19–23].

3. Characterization of j-filters of locally bounded $\underline{K}_2$-algebras

In this section, the notion of j-filters of a locally bounded $\underline{K}_2$-algebra L is presented. Many properties of j-filters will be investigated.

At first, we introduce the definition of locally bounded $\underline{K}_2$-algebras.

Definition 10. *A $\underline{K}_2$-algebra L is called a locally bounded if $L^{\vee}$ is a principal filter of L, that is, there exists $j \in L$ such that $L^{\vee} = [j)$.*

Definition 11. *A filter F of a locally bounded $\underline{K}_2$-algebra L with $L^{\vee} = [j)$ is called a j-filter of L if $j \in F$.*

Example 12. *Consider the algebra $(L; \vee, \wedge, ^{\circ}, 0, 1)$ in* Fig. 1.
We observe that L is a locally bounded $\underline{K}_2$-algebra with

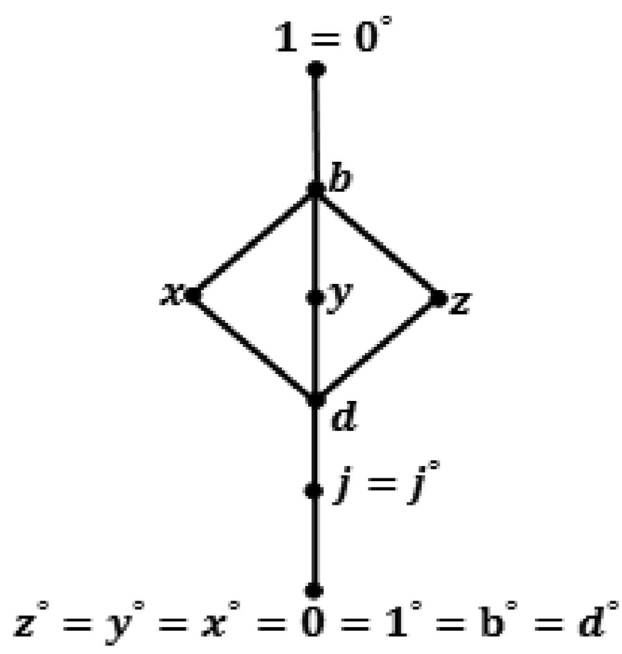
$L^{\vee} = [j) = \{1, b, x, y, z, d, j\}$.

It is clear that L and $L^{\vee}$ are the largest and smallest j-filters of L, respectively. Since j does not belong to each of the filters $[d), [x), [y), [z), [b)$ and $[1)$, then these filters are not j-filters of L.

Definition 13. *Let A be a nonempty subset of a locally bounded*
$\underline{K}_2$-algebra L with $L^{\vee} = [j)$. Define A^{Δ} as follows:

$A^{\Delta} = \{y \in L : y^{\circ\circ} \geq a^{\circ\circ} \wedge j, \text{for some } a \in A\}$.

Lemma 14. *Let L be a locally bounded $\underline{K}_2$-algebra. Let A be a nonempty subset of L which is closed with respect to $\wedge$. Then A^{Δ} is a j-filter of L containing A.*



L

Fig. 1. L is a locally bounded $\underline{K}_2$− algebra with $L^{\vee} = [j)$.

Proof. *Clearly,* $1 \in A^{\Delta}$. *Let* $x, y \in A^{\Delta}$. *Then* $x^{\circ\circ} \geq a_1^{\circ\circ} \wedge j$, *and* $y^{\circ\circ} \geq a_2^{\circ\circ} \wedge j$, *for some* $a_1, a_2 \in A$. *Hence*

$$(x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ} \geq (a_1^{\circ\circ} \wedge j) \wedge (a_2^{\circ\circ} \wedge j) = (a_1^{\circ\circ} \wedge a_2^{\circ\circ}) \wedge j = (a_1 \wedge a_2)^{\circ\circ} \wedge j.$$

Since $a_1 \wedge a_2 \in A$, then $x \wedge y \in A^{\Delta}$. Now, let $y \in A^{\Delta}$ and let $z \geq y, z \in L$. Then $y^{\circ\circ} \geq a^{\circ\circ} \wedge j$, for some $a \in A$. Thus $z^{\circ\circ} \geq y^{\circ\circ} \geq a^{\circ\circ} \wedge j$, for some $a \in A$. Therefore $z \in A^{\Delta}$ and hence A^{Δ} is a filter of L. Since $j^{\circ\circ} \geq j \geq a^{\circ\circ} \wedge j$, for all $a \in A$, then $j \in A^{\Delta}$. Therefore A^{Δ} is a j-filter. Let $x \in A$. Since $x^{\circ\circ} \geq x^{\circ\circ} \wedge j$, then $x \in A^{\Delta}$. Thus $A \subseteq A^{\Delta}$. Then A^{Δ} is a j-filter of L containing A.

This lemma gives the fundamental properties of A^{Δ}.

Lemma 15. *Let A and B be two nonempty subsets of a locally bounded* $\underline{K}_2$-algebra L, *which are closed under* $\wedge$. *Then we have*

(1) $[A) \subseteq A^{\Delta}$ and $L^{\vee} \subseteq A^{\Delta}$,
(2) $A^{\Delta} = [A) \vee L^{\vee}$,
(3) $A \subseteq B \Rightarrow A^{\Delta} \subseteq B^{\Delta}$,
(4) $A^{\Delta\Delta} = A^{\Delta}$,
(5) $[A)\Delta = A^{\Delta}$.

Proof. (1) Let $x \in [A)$. Then

$$x \geq a_1 \wedge a_2 \wedge \ldots \wedge a_n \text{ for some } a_i \in A, i = 1, 2, \ldots, n.$$

Thus $x^{\circ\circ} \geq a_1^{\circ\circ} \wedge a_2^{\circ\circ} \wedge \ldots \wedge a_n^{\circ\circ} \geq (a_1 \wedge a_2 \wedge \ldots \wedge a_n)^{\circ\circ} \wedge j$. Then $x \in A^{\Delta}$ as $a_1 \wedge a_2 \wedge \ldots \wedge a_n \in A$. Hence $[A) \subseteq A^{\Delta}$. Now, let $x \in L^{\vee} = [j)$.
Then $x \geq j$. This implies that $x^{\circ\circ} \geq x \geq j \geq a^{\circ\circ} \wedge j$ for some $a \in A$.
Therefore $x \in A^{\Delta}$ and hence $L^{\vee} \subseteq A^{\Delta}$.
(2) Let $y \in A^{\Delta}$. Then $y^{\circ\circ} \geq a^{\circ\circ} \wedge j$ for some $a \in A$. Since $y = y^{\circ\circ} \wedge (y \vee j)$, then

$$y \geq (a^{\circ\circ} \wedge j) \wedge (y \vee j)$$

$$= a^{\circ\circ} \wedge (j \wedge (y \vee j))$$

$$= a^{\circ\circ} \wedge j, \text{by the absorption identity}$$

$$\geq a \wedge j.$$

This gives that $y \in [a) \vee [j) \subseteq [A) \vee L^{\vee}$. Hence $A^{\Delta} \subseteq [A) \vee L^{\vee}$. Conversely, from (1), $[A) \vee L^{\vee} \subseteq A^{\Delta}$. Therefore $A^{\Delta} = [A) \vee L^{\vee}$.
(3) Let $y \in A^{\Delta}$ and $A \subseteq B$. Then $y^{\circ\circ} \geq a^{\circ\circ} \wedge j$, for some $a \in A \subseteq B$. Thus $a \in B$ and hence $y \in B^{\Delta}$. Then $A^{\Delta} \subseteq B^{\Delta}$.
(4) $A^{\Delta} = \{y \in L : y^{\circ\circ} \geq a^{\circ\circ} \wedge j, \text{for some } a \in A\}$

$$= \{y \in L : y^{\circ\circ} \geq a^{\circ\circ} \wedge j, \text{for some } a \in A \subseteq A^{\Delta}\}, \text{by Lemma 14}$$

$$= A^{\Delta\Delta}.$$

(5) From (2), (3) and (4), we get $[A)\Delta \subseteq A^{\Delta\Delta} = A^{\Delta}$. Conversely, since $A \subseteq [A)$, then again by (3), we get $A^{\Delta} \subseteq [A)\Delta$. Therefore $[A)\Delta = A^{\Delta}$.
Now, the following Theorem presents a characterization of a j-filter of a locally bounded $\underline{K}_2$-algebra L.

Theorem 16. *Let L be a locally bounded* $\underline{K}_2$*-algebra and F be a filter of L.*
Then F is a j-filter of L *if and only if* $F = F^{\Delta}$.

Proof. Clearly, $j \in F$, as F is a j-filter of L. Then By Lemma 13, we have $F \subseteq F^{\Delta}$. Now, let $x \in F^{\Delta}$. Then $x^{\circ\circ} \geq f^{\circ\circ} \wedge j$, for some $f \in F$. Since

$$x = x^{\circ\circ} \wedge (x \vee j) \geq (f^{\circ\circ} \wedge j) \wedge (x \vee j) = f^{\circ\circ} \wedge j \in F,$$

where $f^{\circ\circ}, j \in F$. Then $F^{\Delta} \subseteq F$ and hence $F = F^{\Delta}$. Conversely, let $F = F^{\Delta}$. Since $j \in F^{\Delta} = F$, then F is a j-filter.
Let $F_j(L) = \{F^{\Delta} : F \in F(L)\} = \{F : F \text{ is a } j\text{-filter of } L\}$ be the set of all j-filters of L.

Theorem 17. *Let F and G be filters of a locally bounded* $\underline{K}_2$-algebra L. *Then,*

(1) $(F \vee G)^{\Delta} = F^{\Delta} \vee G^{\Delta}$,
(2) $(F \cap G)^{\Delta} = F^{\Delta} \cap G^{\Delta}$,
(3) $F_j(L)$ is a modular $\{1\}$-sublattice of $F(L)$ and a bounded modular lattice on its own.

Proof. (1) Since $F, G \subseteq F \vee G$, then $F^{\Delta}, G^{\Delta} \subseteq (F \vee G)^{\Delta}$, by Lemma 15 (3). Thus the upper bound of F^{Δ} and G^{Δ} is $(F \vee G)^{\Delta}$. Consider H^{Δ} as another upper bound of F^{Δ} and G^{Δ}, where H is a filter of L. Then $F^{\Delta}, G^{\Delta} \subseteq H^{\Delta}$. Since $F, G \subseteq F^{\Delta}, G^{\Delta}$, then $F, G \subseteq H^{\Delta}$. Hence $F \vee G \subseteq H^{\Delta}$. Then by Lemma 15 (3) and (4), $(F \vee G)^{\Delta} \subseteq H^{\Delta\Delta} = H^{\Delta}$. Therefore $(F \vee G)^{\Delta}$ is the least upper bound of F^{Δ}, G^{Δ} and hence $(F \vee G)^{\Delta} = F^{\Delta} \vee G^{\Delta}$.
(2) We can also show that $(F \cap G)^{\Delta} = F^{\Delta} \cap G^{\Delta}$ by applying a similar method.
(3) From (1) and (2), we obtain that $F_j(L)$ is a $\{1\}$-sublattice of $F(L)$, as $L \in F_j(L)$. Since $F(L)$ is a modular lattice, then $F_j(L)$ is also a modular lattice. It is clear that $L, L^{\vee}$ are the largest and smallest members of $F_j(L)$, respectively. Then $(F_j(L), \vee, \wedge, L^{\vee}, L)$ is a bounded modular lattice on its own.
The following Theorem represents another characterization of a j-filter of a locally bounded $\underline{K}_2$-algebra L.

Theorem 18. *Let F be a proper filter of a locally bounded* $\underline{K}_2$*-algebra L. Then we have the equivalent conditions*

(1) F is a j-filter,
(2) $x \vee j \in F$, for all $x \in L$,
(3) $L^{\vee} \subseteq F$.

Proof. (1)$\Rightarrow$(2): Let F be a j-filter of L. Then $j\in F$. Since $x\vee j\geq j\in F$, for all $x\in L$, then $x\vee j\in F$.
(2)$\Rightarrow$(3): Let $x\in L^{\vee}$. Then $x\geq j\in L^{\vee}$. Thus we have $x=x\vee j\in F$, by (2). Then $x\in F$ and hence $L^{\vee}\subseteq F$.
(3)$\Rightarrow$(1): Since $j\in L^{\vee}\subseteq F$, then F is a j-filter of L.

4. Principal j-filters of locally bounded $\underline{K}_2$-algebras

In this section, we define and characterize the principal j-filters of a locally bounded $\underline{K}_2$-algebra L. Then we investigate the basic properties of such filters.

For any element x of a locally bounded $\underline{K}_2$-algebra L with $L^{\vee}=[j)$, we write $(x)^{\Delta}$ instead of $(\{x\})^{\Delta}$. We observe that $(x)^{\Delta}=\{y\in L: y^{\circ\circ}\geq x^{\circ\circ}\wedge j\}$.

It is clear that $(1)^{\Delta}=L^{\vee}$ and $(0)^{\Delta}=L$ are the smallest and largest j-filters of L, respectively.

Lemma 19. *Let L be a locally bounded $\underline{K}_2$-algebra. Then $(x)^{\Delta}$ is a principal j-filter containing x, precisely* $(x)^{\Delta}=[x^{\circ\circ}\wedge j)=[x^{\circ\circ})\vee L^{\vee}$.

Proof. According to Lemma 14, $(x)^{\Delta}$ is a j-filter of L containing x. Now, we show that $(x)^{\Delta}=[x^{\circ\circ}\wedge j)$. Let $y\in(x)^{\Delta}$. Then $y^{\circ\circ}\geq x^{\circ\circ}\wedge j$. Thus we have

$$y=y^{\circ\circ}\wedge(y\vee j)\geq x^{\circ\circ}\wedge j\wedge(y\vee j)=x^{\circ\circ}\wedge j, \text{as } j\wedge(y\vee j)=j.$$

Hence $y\in[x^{\circ\circ}\wedge j)$. Therefore $(x)^{\Delta}\subseteq[x^{\circ\circ}\wedge j)$. Conversely, let $y\in[x^{\circ\circ}\wedge j)$. Then $y^{\circ\circ}\geq y\geq x^{\circ\circ}\wedge j$ implies $y\in(x)^{\Delta}$. Thus $[x^{\circ\circ}\wedge j)\subseteq(x)^{\Delta}$. Consequently, $(x)^{\Delta}=[x^{\circ\circ}\wedge j)$.

Theorem 20. *Let x and y be two elements of a locally bounded $\underline{K}_2$-algebra L. Then,*

(1) $[x)\subseteq(x)^{\Delta}$,
(2) $(x)^{\Delta}=(x^{\circ\circ})^{\Delta}$,
(3) $x^{\circ\circ}=y^{\circ\circ}\Rightarrow(x)^{\Delta}=(y)^{\Delta}$ but the converse is not true,
(4) $x\leq y\Leftrightarrow(y)^{\Delta}\subseteq(x)^{\Delta}$,
(5) $x\in(y)^{\Delta}\Leftrightarrow(x)^{\Delta}\subseteq(y)^{\Delta}$.

Proof. (1) Let $y\in[x)$. Then $y\geq x$ implies $y^{\circ\circ}\geq x^{\circ\circ}\geq x^{\circ\circ}\wedge j$. Therefore $y\in(x)^{\Delta}$ and hence $[x)\subseteq(x)^{\Delta}$.
(2) Using the fact that $x^{\circ\circ\circ\circ}=x^{\circ\circ}$, we get

$$(x^{\circ\circ})^{\Delta}=\{y\in L:\ y^{\circ\circ}\geq x^{\circ\circ\circ\circ}\wedge j\}$$

$$=\{y\in L:\ y^{\circ\circ}\geq x^{\circ\circ}\wedge j\}$$

$$=(x)^{\Delta}.$$

(3) Let $x^{\circ\circ}=y^{\circ\circ}$. Then,

$$(x)^{\Delta}=\{a\in L: a^{\circ\circ}\geq x^{\circ\circ}\wedge j\}$$

$$=\{a\in L:\ a^{\circ\circ}\geq y^{\circ\circ}\wedge j\}$$

$$=(y)^{\Delta}.$$

In Example 12, $(1)^{\Delta}=(j)^{\Delta}=[j)$ but $1^{\circ\circ}\neq j^{\circ\circ}$. So the converse is not true.
(4) Let $x\leq y$. Then $x^{\circ\circ}\leq y^{\circ\circ}$ and $x^{\circ\circ}\wedge j\leq y^{\circ\circ}\wedge j$. Let $a\in(y)^{\Delta}$. Then $a^{\circ\circ}\geq y^{\circ\circ}\wedge j\geq x^{\circ\circ}\wedge j$. Hence $a\in(x)^{\Delta}$. Thus $(y)^{\Delta}\subseteq(x)^{\Delta}$.
Conversely, let $(y)^{\Delta}\subseteq(x)^{\Delta}$. Then,

$$y\in(y)^{\Delta}\subseteq(x)^{\Delta}.$$

$$\Rightarrow y^{\circ\circ}\geq x^{\circ\circ}\wedge j$$

$$\Rightarrow y=y^{\circ\circ}\wedge(y\vee j)\geq x^{\circ\circ}\wedge j\wedge(y\vee j)$$

$$\Rightarrow y\geq x^{\circ\circ}\wedge j, \text{by the absorption identity}$$

$$\Rightarrow y\geq x^{\circ\circ}\wedge j\geq x\wedge j, \text{as } x^{\circ\circ}\geq x.$$

We claim that $x\leq j$. If $j\leq x$, then $(x)^{\Delta}$ is a proper subset of $(j)^{\Delta}=L^{\vee}$, which is a contradiction as $L^{\vee}$ is the smallset j-filter of L. Therefore $x\leq y$.
(5) Let $a\in(x)^{\Delta}$ and $x\in(y)^{\Delta}$. Then $a^{\circ\circ}\geq x^{\circ\circ}\wedge j$ and $x^{\circ\circ}\geq y^{\circ\circ}\wedge j$. It follows that $a^{\circ\circ}\geq y^{\circ\circ}\wedge j$. Then $a\in(y)^{\Delta}$ and hence $(x)^{\Delta}\subseteq(y)^{\Delta}$. Conversely, let $(x)^{\Delta}\subseteq(y)^{\Delta}$. Then by Lemma 19, $x\in(x)^{\Delta}\subseteq(y)^{\Delta}$.

Theorem 21. *Let L be a locally bounded $\underline{K}_2$*-algebra. Then $(x)^{\Delta}$ *represents every principal* j-filter of L, for some $x\in L$.

Proof. Let $F=[x)$ be a principal j-filter of L. Let $y\in[x)$. Then $y\geq x$ implies $y^{\circ\circ}\geq x^{\circ\circ}\geq x^{\circ\circ}\wedge j$. Thus $y\in(x)^{\Delta}$. Hence $F\subseteq(x)^{\Delta}$. Conversely, let $y\in(x)^{\Delta}$. Then $y\geq x^{\circ\circ}\wedge j\geq x\wedge j$. Hence $y\in[x)\vee L^{\vee}=[x)$, as $L^{\vee}=[j)$ is the smallest j-filter of L. Then $y\in[x)$. Therefore $(x)^{\Delta}\subseteq[x)$. Then every principal j-filter $[x)$ can be expressed as $(x)^{\Delta}$.
The set of all principal j-filters of L is denote by $F_j^p(L)=\{(x)^{\Delta}: x\in L\}$.

Theorem 22. *Let x and y be two elements of a locally bounded $\underline{K}_2$*-algebra *L*. Then,

(1) $(x\wedge y)^{\Delta}=(x)^{\Delta}\vee(y)^{\Delta}$,
(2) $(x\vee y)^{\Delta}=(x)^{\Delta}\cap(y)^{\Delta}$,
(3) $F_j^p(L)$ is a bounded sublattice of $F_j(L)$.

Proof. (1) Let $x, y \in L$. Then,

$$(x \wedge y)^{\Delta} = [(x \wedge y)^{\circ\circ} \wedge j)$$

$$= [x^{\circ\circ} \wedge y^{\circ\circ} \wedge j)$$

$$= [(x^{\circ\circ} \wedge j) \wedge (y^{\circ\circ} \wedge j))$$

$$= [x^{\circ\circ} \wedge j) \vee [y^{\circ\circ} \wedge j)$$

$$= (x)^{\Delta} \vee (y)^{\Delta}.$$

(2) Since $x, y \leq x \vee y$, then by Theorem 20, $(x \vee y)^{\Delta} \subseteq (x)^{\Delta}, (y)^{\Delta}$. Thus $(x \vee y)^{\Delta} \subseteq (x)^{\Delta} \cap (y)^{\Delta}$. Conversely, let $a \in (x)^{\Delta} \cap (y)^{\Delta}$. Then $a^{\circ\circ} \geq x^{\circ\circ} \wedge j$ and $a^{\circ\circ} \geq y^{\circ\circ} \wedge j$. Hence,

$$a^{\circ\circ} \geq (x^{\circ\circ} \wedge j) \vee (y^{\circ\circ} \wedge j) = (x \vee y)^{\circ\circ} \wedge j.$$

Then $a \in (x \vee y)^{\Delta}$. Thus $(x)^{\Delta} \cap (y)^{\Delta} \subseteq (x \vee y)^{\Delta}$.
Therefore $(x \vee y)^{\Delta} = (x)^{\Delta} \cap (y)^{\Delta}$.
(3) We observe that $(0)^{\Delta} = L$ and $(1)^{\Delta} = L^{\vee}$ are the greatest and smallest principal j-filters of L, respectively. Then by (1) and (2), $(F_j^p(L); \vee, \wedge, L^{\vee}, L)$ is a bounded sublattice of $F_j(L)$.
The following lemma shows that the element j is a distributive element of a locally bounded $\underline{K}_2$-algebra L, which is a useful property.

Lemma 23. *Let L be a locally bounded $\underline{K}_2$-algebra. Then*

(1) $(a \vee b) \wedge j = (a \wedge j) \vee (b \wedge j)$, for all $a, b \in L^{\circ\circ}$,
(2) $(x \vee y) \wedge j = (x \wedge j) \vee (y \wedge j)$, for all $x, y \in L$.

Proof. (1) Let $a, b \in L^{\circ\circ}$. Then $a^{\circ\circ} = a, b^{\circ\circ} = \mathrm{b}$ and hence
$(a \vee b)^{\circ\circ} = a \vee b$. Using Theorem 22 (2), we have

$$(a \vee b)^{\Delta} = (a)^{\Delta} \cap (b)^{\Delta},$$

$$\Rightarrow [(a \vee b)^{\circ\circ} \wedge j) = [a^{\circ\circ} \wedge j) \cap [b^{\circ\circ} \wedge \mathrm{j})$$

$$\Rightarrow [(a \vee b) \wedge j) = [(a \wedge j) \vee (b \wedge j)).$$

Then $(a \vee b) \wedge j = (a \wedge j) \vee (b \wedge j)$.
(2) Since $x = x^{\circ\circ} \wedge (x \vee j), y = y^{\circ\circ} \wedge (y \vee j)$, then we get

$$(x \wedge j) \vee (y \wedge j) = ((x^{\circ\circ} \wedge (x \vee j)) \wedge j) \vee ((y^{\circ\circ} \wedge (y \vee j)) \wedge j)$$

$$= (x^{\circ\circ} \wedge j) \vee (y^{\circ\circ} \wedge j), \text{by the absorption identity}$$

$$= (x^{\circ\circ} \vee y^{\circ\circ}) \wedge j, \text{by}(1)$$

$$= (x \vee y)^{\circ\circ} \wedge j$$

$$= (x \vee y)^{\circ\circ} \wedge ((x \vee y) \vee j) \wedge j, \text{as}\, j \leq (x \vee y) \vee j$$

$$= \{(x \vee y)^{\circ\circ} \wedge ((x \vee y) \vee j)\} \wedge j$$

$$= (x \vee y) \wedge j, \text{where}\, x \vee y = (x \vee y)^{\circ\circ} \wedge ((x \vee y) \vee j).$$

Therefore j is a distributive element of L.

Lemma 24. *Let L be a locally bounded $\underline{K}_2$-algebra. Then,*

(1) $(x)^{\Delta} = L^{\vee} \Leftrightarrow x \in L^{\vee}$,
(2) $(x)^{\Delta} = [x) \Leftrightarrow x \in L^{\wedge}$,
(3) $(x)^{\Delta} = L \Leftrightarrow x = 0$

Proof. (1) Let $(x)^{\Delta} = L^{\vee}$. Then $[x^{\circ\circ} \wedge j) = [j)$. Implies $x^{\circ\circ} \wedge j = j$.
Then $x^{\circ\circ} \geq j$ and hence $x^{\circ\circ} \in [j) = L^{\vee}$. Now, $x^{\circ\circ} \in L^{\vee}$, $x \vee j \in L^{\vee}$ imply
$x = x^{\circ\circ} \wedge (x \vee j) \in L^{\vee}$. Conversely, let $x \in L^{\vee} = [j)$. Then $x^{\circ\circ} \geq x \geq j$. Thus

$$(x)^{\Delta} = [x^{\circ\circ} \wedge j)$$

$$= [j) = L^{\vee}, \text{as}\, j \leq x^{\circ\circ}.$$

(2) Let $(x)\Delta = [x)$. Since $x = x^{\circ\circ} \wedge (x \vee x^{\circ})$, then,

$$(x)^{\Delta} = [x) \Rightarrow [x^{\circ\circ} \wedge j) = [x)$$

$$\Rightarrow x^{\circ\circ} \wedge j = x$$

$$\Rightarrow x^{\circ\circ} \wedge j \wedge (x \vee x^{\circ}) = x \wedge (x \vee x^{\circ})$$

$$\Rightarrow x \wedge j = x, \text{by the absorption identity}.$$

Thus $x \leq j$ and hence $x \in L^{\wedge}$. Conversely, at first we need to prove that $L^{\wedge} \subseteq L^{\circ\circ}$. Let $x \in L^{\wedge}$. Then $x \wedge x^{\circ} = x$. Implies $(x \wedge x^{\circ})^{\circ\circ} = x^{\circ\circ}$. Therefore,
$x = x \wedge x^{\circ} = x^{\circ\circ} \wedge x^{\circ\circ\circ} = x^{\circ\circ}$. Then $x = x^{\circ\circ} \in L^{\circ\circ}$. Now,

$$(x)^{\Delta} = [x^{\circ\circ} \wedge j)$$

$$= [x \wedge j)$$

$$= [x), \text{as}\, x \leq j, \text{by Remark 5}.$$

(3) Let $x = 0$. Then $x^{\circ\circ} = 0$ and $(x)^{\Delta} = [x^{\circ\circ} \wedge j) = [0) = L$. Conversely, let $(x)^{\Delta} = L$. Then $[x^{\circ\circ} \wedge j) = L = [0)$ implies $x^{\circ\circ} \wedge j = 0$. Since $j \neq 0$, then $x^{\circ\circ} = 0$. Thus $x = 0$.
Now, we give a characterization of a j-filter of a locally bounded $\underline{K}_2$-algebra L via principal j-filters of L.

Theorem 25. *Let F be a filter of a locally bounded $\underline{K}_2$-algebra L. Then F is a j-filter of L if and only if* $F = \bigcup_{x \in F} (x)^{\Delta}$.

Proof. Let F be a j-filter and $y \in F$. Since $y^{\circ\circ} \geq y^{\circ\circ} \wedge j$, then $y \in (y)^{\Delta} \subseteq \underset{x \in F}{\cup} (x)^{\Delta}$. Thus $F \subseteq \underset{x \in F}{\cup} (x)^{\Delta}$. On the other hand, let $y \in \underset{x \in F}{\cup} (x)^{\Delta}$. Then $y \in (z)^{\Delta}$ for some $z \in F$. Hence $y^{\circ\circ} \geq z^{\circ\circ} \wedge j \in F$. This implies that

$$y = y^{\circ\circ} \wedge (y \vee j) \geq z^{\circ\circ} \wedge j \wedge (y \vee j)$$

$$y \geq z^{\circ\circ} \wedge j \in F, \text{by the absorption identity.}$$

Then $y \in F$. Therefore $\underset{x \in F}{\cup} (x)^{\Delta} \subseteq F$ and hence $F = \underset{x \in F}{\cup} (x)^{\Delta}$.

Conversely, since $j \in (x)^{\Delta} \subseteq \underset{x \in F}{\cup} (x)^{\Delta} = F$, then F is a j-filter of L.

Theorem 26. *The class $F_j^p(L)$ of all principal j-filters of a locally bounded $\underline{K}_2$-algebra L forms a Kleene algebra.*

Proof. By Theorem 22 (3), $(F_j^p(L); \vee, \wedge, L^{\vee}, L)$ is a bounded lattice. Now, we show that $F_j^p(L)$ is a distributive lattice. Let $(x)^{\Delta}, (y)^{\Delta}, (z)^{\Delta} \in F_j^p(L)$. Then,

$$(x)^{\Delta} \cap [(y)^{\Delta} \vee (z)^{\Delta}] = (x)^{\Delta} \cap (y \wedge z)^{\Delta}$$

$$= [x^{\circ\circ} \wedge j) \cap [(y \wedge z)^{\circ\circ} \wedge j)$$

$$= [(x^{\circ\circ} \wedge j) \vee ((y \wedge z)^{\circ\circ} \wedge j))$$

$$= [(x^{\circ\circ} \vee (y \wedge z)^{\circ\circ}) \wedge j), \text{by Lemma 23}$$

$$= \left[(x^{\circ\circ} \vee y^{\circ\circ}) \wedge (x^{\circ\circ} \vee z^{\circ\circ}) \wedge j\right), \text{by distributivity of } L^{\circ\circ}$$

$$= [(x \vee y)^{\circ\circ} \wedge j) \vee [(x \vee z)^{\circ\circ} \wedge j)$$

$$= (x \vee y)^{\Delta} \vee (x \vee z)^{\Delta}$$

$$= [(x)^{\Delta} \cap (y)^{\Delta}] \vee [(x)^{\Delta} \cap (z)^{\Delta}], \text{by Theorem 22 (2).}$$

Hence $F_j^p(L)$ is a distributive lattice. To show that $F_j^p(L)$ is a Kleene algebra, define a unary operation $^-$ on $F_j^p(L)$ by $\overline{(x)^{\Delta}} = (x^{\circ})^{\Delta}$, for all $(x)^{\Delta} \in F_j^p(L)$. Then we have

$$\overline{\overline{(x)^{\Delta}}} = (x^{\circ\circ})^{\Delta} = (x)^{\Delta}, \overline{(1)^{\Delta}} = (1^{\circ})^{\Delta} = (0)^{\Delta},$$
$$\text{and } \overline{(x)^{\Delta} \cap (y)^{\Delta}} = \overline{(x \vee y)^{\Delta}}, \text{by}$$

Theorem 22(2)

$$= ((x \vee y)^{\circ})^{\Delta}$$

$$= (x^{\circ} \wedge y^{\circ})^{\Delta}$$

$$= (x^{\circ})^{\Delta} \vee (y^{\circ})^{\Delta}, \text{by Theorem 22(1)}$$

$$= \overline{(x)^{\Delta}} \vee \overline{(y)^{\Delta}}.$$

Since $y \wedge y^{\circ} \leq x \vee x^{\circ}$, then we have

$$(x)^{\Delta} \cap \overline{(x)^{\Delta}} = (x)^{\Delta} \cap (x^{\circ})^{\Delta}$$

$$= (x \vee x^{\circ})^{\Delta} \subseteq (y \wedge y^{\circ})^{\Delta}, \text{by Theorem 20(4)}$$

$$= (y)^{\Delta} \vee (y^{\circ})^{\Delta}, \text{by Theorem 22(1)}$$

$$= (y)^{\Delta} \vee \overline{(y)^{\Delta}}.$$

Hence $(F_j^p(L); \vee, \wedge, ^-, L^{\vee}, L)$ is a Kleene algebra.

Now, we construct an example to clarify the above results.

Example 27. *Consider the locally bounded $\underline{K}_2$-algebra L as in* Fig. 2.

We observe that $L^{\vee} = [j) = \{1, x, y, z, e, c, j\}$ and $L^{\circ\circ} = \{0, a, b, c, d, 1\}$.

A description of the lattice $F_j^p(L)$ is given in Fig. 3.

It is clear that $(F_j^p(L), ^-)$ is a Kleene algebra.

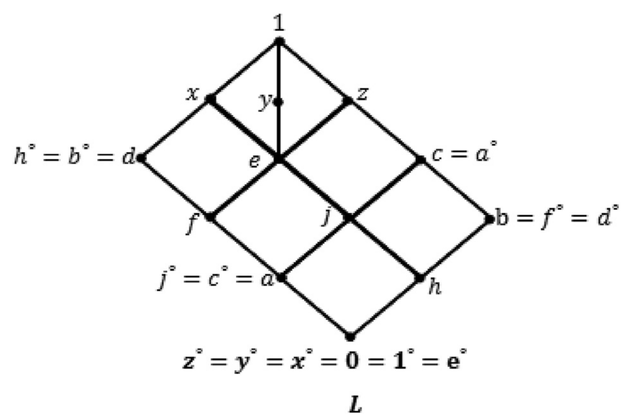


Fig. 2. L is a locally bounded $\underline{K}_2-$ algebra.

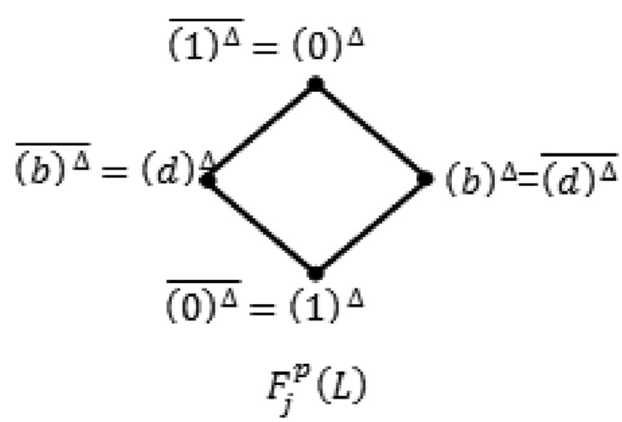


Fig. 3. $F_j^p(L)$ is a Kleene algebra.

Define a relation ψ on a locally bounded $\underline{K}_2$-algebra L by:

$$(x,y)\in\psi\Leftrightarrow x^{\circ\circ}=y^{\circ\circ}\Leftrightarrow x^{\circ}=y^{\circ}.$$

Theorem 28. *Let L be a locally bounded $\underline{K}_2$-algebra. Then we have*

(1) ψ is a congruence relation on L,
(2) $[x]_\psi=[x^{\circ\circ}]_\psi$, where $[x]_\psi=\{a\in L: a^{\circ\circ}=x^{\circ\circ}\}$ is the congruence class of an element x of L,
(3) $x^{\circ\circ}=\max[x]_\psi$,
(4) $[1]_\psi=D(L), [0]_\psi=\{0\}$,
(5) L/ψ is a Kleene algebra,
(6) $L^{\circ\circ}\cong L/\psi$.

Proof. (1) Clearly ψ is an equivalence relation on L. Let $(a,b),(c,d)\in\psi$. Then $a^{\circ\circ}=b^{\circ\circ}$ and $c^{\circ\circ}=d^{\circ\circ}$. Thus we have

$$(a\vee c)^{\circ\circ}=a^{\circ\circ}\vee c^{\circ\circ}$$

$$=b^{\circ\circ}\vee d^{\circ\circ}$$

$$=(b\vee d)^{\circ\circ},$$

and

$$(a\wedge c)^{\circ\circ}=a^{\circ\circ}\wedge c^{\circ\circ}$$

$$=b^{\circ\circ}\wedge d^{\circ\circ}$$

$$=(b\wedge d)^{\circ\circ}.$$

Then $(a\vee c,b\vee d),(a\wedge c,b\wedge d)\in\psi$. Therefore ψ is a lattice congruence on L. Let $(x,y)\in\psi$. Then $x^{\circ\circ}=y^{\circ\circ}$ implies $x^{\circ\circ\circ}=y^{\circ\circ\circ}$. Thus $(x^\circ,y^\circ)\in\psi$, and hence ψ is a congruence relation on L.
(2) Since $[x]_\psi=\{a\in L:\ a^{\circ\circ}=x^{\circ\circ}\}$, then

$$[x^{\circ\circ}]_\psi=\{a\in L: a^{\circ\circ}=x^{\circ\circ\circ\circ}=x^{\circ\circ}\}=[x]_\psi$$

(3) Let $a\in[x]_\psi$. Then $a\le a^{\circ\circ}=x^{\circ\circ}$, for all $a\in[x]_\psi$. Thus $x^{\circ\circ}=\max[x]_\psi$.
(4) Clearly, $[1]_\psi=\{a\in L: a^{\circ\circ}=1^{\circ\circ}=1\}=D(L)$ and

$$[0]_\psi=\{a\in L: a^{\circ\circ}=0^{\circ\circ}=0\}=\{0\}.$$

(5) Consider the quotient set $L/\psi=\{[x]_\psi: x\in L\}$. It is known that
$(L/\psi;\vee,\wedge,[0]_\psi,[1]_\psi)$ is a bounded lattice, where

$$[x]_\psi\vee[y]_\psi=[x\vee y]_\psi,$$

$$[x]_\psi\wedge[y]_\psi=[x\wedge y]_\psi.$$

Define a unary operation $\blacksquare$ on L/ψ by $[x]_\psi^{\blacksquare}=[x^\circ]_\psi$, for every $[x]_\psi\in L/\psi$. Then we have

$$\left([x]_\psi\wedge[y]_\psi\right)^{\blacksquare}=[x\wedge y]_\psi^{\blacksquare}$$

$$=[(x\wedge y)^\circ]_\psi$$

$$=[x^\circ\vee y^\circ]_\psi$$

$$=[x^\circ]_\psi\vee[y^\circ]_\psi$$

$$=[x]_\psi^{\blacksquare}\vee[y]_\psi^{\blacksquare},$$

$[x]_\psi^{\blacksquare\blacksquare}=[x^{\circ\circ}]_\psi=[x]_\psi$ and $[1]_\psi^{\blacksquare}=[1^\circ]_\psi=[0]_\psi$.

Also,

$$[x]_\psi\wedge[x]_\psi^{\blacksquare}=[x\wedge x^\circ]_\psi\subseteq[y\vee y^\circ]=[y]_\psi\vee[y]_\psi^{\blacksquare}.$$

Therefore $(L/\psi;^{\blacksquare})$ is a Kleene algebra.
(6) Define a map $f: L^{\circ\circ}\to L/\psi$ by $f(a)=[a]_\psi$, for all $a\in L^{\circ\circ}$.
One can show that f is a $(0,1)$ lattice homomorphism. Since

$$f(a^\circ)=[a^\circ]_\psi=[a]_\psi^{\blacksquare}=\left(f(a)\right)^{\blacksquare},$$

then f is a homomorphism. To show that f is an injective map, let $f(a)=f(b)$. Then $[a]_\psi=[b]_\psi$ implies $a^{\circ\circ}=b^{\circ\circ}$. Thus $a=b$, as $a,b\in L^{\circ\circ}$. For every $[a]_\psi\in L/\psi$ we have $[a]_\psi=[a^{\circ\circ}]_\psi=f(a^{\circ\circ}), a^{\circ\circ}\in L^{\circ\circ}$. Then f is a surjective map and hence f is an isomorphism of the Kleene algebras $L^{\circ\circ}$ and L/ψ.

Lemma 29. *Let x and y be any two elements of a locally bounded $\underline{K}_2$-algebra L. Then,*

(1) $[x]_\psi=[y]_\psi\Rightarrow(x)^\Delta=(y)^\Delta$,
(2) $[[x]_\psi)=\{y\in L: y^{\circ\circ}\ge x^{\circ\circ}\}$,
(3) $[[x]_\psi)\subseteq(x)^\Delta$,
(4) $[[x]_\psi)=L^\vee$, if $x\in(L^\vee-D(L))$,
(5) $[[x]_\psi)$ is not a j-filter, if $x\in D(L)$.

Proof. (1) Let $[x]_\psi=[y]_\psi$. Then $x^{\circ\circ}=y^{\circ\circ}$. Then by Theorem 20 (3), we get $(x)^\Delta=(y)^\Delta$.
(2) Since $[x]_\psi=\{a\in L: a^{\circ\circ}=x^{\circ\circ}\}$, then

$$\left[[x]_\psi\right)=\{y\in L: y\ge a_1\wedge a_2\wedge\ldots\wedge a_n, a_i\in[x]_\psi, i=1,\ldots,n\}$$

$$=\{y\in L: y^{\circ\circ}\ge(a_1\wedge a_2\wedge\ldots\wedge a_n)^{\circ\circ}\}$$

$$=\{y\in L: y^{\circ\circ}\ge x^{\circ\circ}\}, \text{as } a_1^{\circ\circ}=\ldots=a_n^{\circ\circ}=x^{\circ\circ}.$$

(3) Let $y\in[[x]_\psi)$. Then $y^{\circ\circ}\ge x^{\circ\circ}\ge x^{\circ\circ}\wedge j$ implies

$$y=y^{\circ\circ}\wedge(y\vee j)\ge(x^{\circ\circ}\wedge j)\wedge(y\vee j)\ge x^{\circ\circ}\wedge j.$$

Hence $y\in[x^{\circ\circ}\wedge j)=(x)^\Delta$. Therefore $[[x]_\psi)\subseteq(x)^\Delta$.
(4) Let $x\in(L^\vee-D(L))$. Then $x\in L^\vee$ and $x\notin D(L)$. Now, we have

$[[x]_\psi) = \{y \in L : y^{\circ\circ} \geq x^{\circ\circ}\}$

$= \{y \in L : y^{\circ\circ} \geq x^{\circ\circ} \geq x \geq j\}$, as $x \in L^{\vee}$

$= \{y \in L : y = y^{\circ\circ} \wedge (y \vee j) \geq j \wedge (y \vee j) = j\}$

$= \{y \in L : y \geq j\} = [j) = L^{\vee}$.

(5) Let $x \in D(L)$. Then $x^{\circ} = 0$ and

$[[x]_\psi) = \{y \in L : y^{\circ\circ} \geq x^{\circ\circ}\}$

$= \{y \in L : y^{\circ\circ} \geq 1\}$

$= \{y \in L : y^{\circ\circ} = 1\}$

$= D(L)$.

Since $j \notin D(L)$, then $[[x]_\psi)$ is not a j-filter of L.

5. j-lattice congruences of a locally bounded $\underline{K}_2$-algebra

In this section, we investigate the relationship between the lattice congruence relations and the j-filters of a locally bounded $\underline{K}_2$-algebra L.

Definition 30. A lattice congruence θ on a locally bounded $\underline{K}_2$-algebra L is called a j-lattice congruence on L if $\in Coker\ \theta$.

Lemma 31. *Let θ be a j-lattice congruence on L. Then $Coker\ \theta$ is a j-filter of L.*

Define a binary relation θ_j on a locally bounded $\underline{K}_2$-algebra L by:

$(x, y) \in \theta_j \Leftrightarrow x \wedge j = y \wedge j$, where $x, y \in L$.

Theorem 32. *Let L be a locally bounded $\underline{K}_2$-algebra. Then we have*

(1) θ_j is a j-lattice congruence with $Co\ker\ \theta_j = L^{\vee}$,
(2) $[x]_{\theta_j} = [x^{\circ\circ}]_{\theta_j}$, where $[x]_{\theta_j}$ is the congruence class of x modulo θ_j,
(3) L/θ_j is a Kleene algebra .

Proof. (1) Clearly, θ_j is a lattice congruence. Now, we prove that $Co\ker\ \theta_j = L^{\vee}$.

$Co\ker \theta_j = \{x \in L : (x, 1) \in \theta_j\}$

$= \{x \in L : x \wedge j = 1 \wedge j\}$

$= \{x \in L : x \wedge j = j\}$

$= \{x \in L : x \geq j\}$

$= [j) = L^{\vee}$.

(2) Since $x = x^{\circ\circ} \wedge (x \vee j)$, then $x \wedge j = x^{\circ\circ} \wedge (x \vee j) \wedge j = x^{\circ\circ} \wedge j$. This implies $(x, x^{\circ\circ}) \in \theta_j$ and hence $[x]_{\theta_j} = [x^{\circ\circ}]_{\theta_j}$.

(3) It is observed that $L/\theta_j = \{[x]_{\theta_j} : x \in L\}$ and $[x]_{\theta_j} = \{y \in L : (y, x) \in \theta_j\}$ is the congruence class of x modulo θ_j. It is known that $(L/\theta_j; \vee, \wedge, [0]_{\theta_j}, [1]_{\theta_j})$ is a bounded lattice, where

$$[x]_{\theta_j} \vee [y]_{\theta_j} = [x \vee y]_{\theta_j}, [x]_{\theta_j} \wedge [y]_{\theta_j} = [x \wedge y]_{\theta_j} \text{ and } [1]_{\theta_j} = L^{\vee}, [0]_{\theta_j} = \{0\}.$$

We show that L/θ_j is a distributive lattice. Let $[x]_{\theta_j}, [y]_{\theta_j}, [z]_{\theta_j} \in L/\theta_j$, we have $[x]_{\theta_j} \wedge ([y]_{\theta_j} \vee [z]_{\theta_j}) = [x^{\circ\circ}]_{\theta_j} \wedge ([y^{\circ\circ}]_{\theta_j} \vee [z^{\circ\circ}]_{\theta_j})$, by (2)

$= [x^{\circ\circ}]_{\theta_j} \wedge [y^{\circ\circ} \vee z^{\circ\circ}]_{\theta_j}$

$= [x^{\circ\circ} \wedge (y^{\circ\circ} \vee z^{\circ\circ})]_{\theta_j}$

$= [(x^{\circ\circ} \wedge y^{\circ\circ}) \vee (x^{\circ\circ} \wedge z^{\circ\circ})]_{\theta_j}$, by distributivity of $L^{\circ\circ}$

$= [x^{\circ\circ} \wedge y^{\circ\circ}]_{\theta_j} \vee [x^{\circ\circ} \wedge z^{\circ\circ}]_{\theta_j}$

$= \left([x^{\circ\circ}]_{\theta_j} \wedge [y^{\circ\circ}]_{\theta_j}\right) \vee \left([x^{\circ\circ}]_{\theta_j} \wedge [z^{\circ\circ}]_{\theta_j}\right)$

$= \left([x]_{\theta_j} \wedge [y]_{\theta_j}\right) \vee \left([x]_{\theta_j} \wedge [z]_{\theta_j}\right)$, by (2).

Then L/θ_j is a bounded distributive lattice.

Define an operation $^{\diamond}$ on L/θ_j by $[x]^{\diamond}_{\theta_j} = [x^{\circ}]_{\theta_j}$, for every $[x]_{\theta_j} \in L/\theta_j$. Then we have $[x]^{\diamond\diamond}_{\theta_j} = [x^{\circ\circ}]_{\theta_j} = [x]_{\theta_j}$,

$\left([x]_{\theta_j} \wedge [y]_{\theta_j}\right)^{\diamond} = [x \wedge y]^{\diamond}_{\theta_j}$

$= [(x \wedge y)^{\circ}]_{\theta_j}$

$= [x^{\circ} \vee y^{\circ}]_{\theta_j}$

$= [x^{\circ}]_{\theta_j} \vee [y^{\circ}]_{\theta_j}$

$= [x]^{\diamond}_{\theta_j} \vee [y]^{\diamond}_{\theta_j}$,

and

$[x]_{\theta_j} \wedge [x]^{\diamond}_{\theta_j} = [x \wedge x^{\circ}]_{\theta_j} \leq [y \vee y^{\circ}] = [y]_{\theta_j} \vee [y]^{\diamond}_{\theta_j}$.

Therefore $(L/\theta_j; {}^{\circ})$ is a Kleene algebra.
Suppose that F is a j-filter of a locally bounded $\underline{K}_2$-algebra L. Define a relation θ_F on L as follows:

$$(x,y)\in\theta_F \Leftrightarrow x^{\circ\circ}\wedge f^{\circ\circ}\wedge j = y^{\circ\circ}\wedge f^{\circ\circ}\wedge j, \text{for some } f\in F.$$

Theorem 33. *Let F and G be j-filters of a locally bounded $\underline{K}_2$-algebra L. Then*

(1) θ_F is a j-lattice congruence on L with $Co\ker\theta_F = F$,
(2) $F\subseteq G \Leftrightarrow \theta_F\subseteq\theta_G$.

Proof. (1) It is clear that θ_F is an equivalence relation on L. Let $(a,b),(c,d)\in\theta_F$. Then $a^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j = b^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j$ and $c^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j = d^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j$, for some $f_1, f_2\in F$. Thus we have

$$(a\vee c)^{\circ\circ}\wedge(f_1\wedge f_2)^{\circ\circ}\wedge j = (a^{\circ\circ}\vee c^{\circ\circ})\wedge(f_1^{\circ\circ}\wedge f_2^{\circ\circ})\wedge j$$

$$= \left[(a^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ})\vee(c^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ})\right]\wedge j, \text{by distributivity of } L^{\circ\circ}$$

$$= (a^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j)\vee(c^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j), \text{by Lemma 23}$$

$$= (b^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j)\vee(d^{\circ\circ}\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j)$$

$$= (b^{\circ\circ}\vee d^{\circ\circ})\wedge(f_1^{\circ\circ}\wedge f_2^{\circ\circ})\wedge j$$

$$= (b\vee d)^{\circ\circ}\wedge(f_1\wedge f_2)^{\circ\circ}\wedge j,$$

where $f_1\wedge f_2\in F$, and

$$(a\wedge c)^{\circ\circ}\wedge(f_1\wedge f_2)^{\circ\circ}\wedge j = (a^{\circ\circ}\wedge c^{\circ\circ})\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j$$

$$= (a^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j)\wedge(c^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j)$$

$$= (b^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j)\wedge(d^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j)$$

$$= (b^{\circ\circ}\wedge d^{\circ\circ})\wedge f_1^{\circ\circ}\wedge f_2^{\circ\circ}\wedge j$$

$$= (b\wedge d)^{\circ\circ}\wedge(f_1\wedge f_2)^{\circ\circ}\wedge j.$$

Hence $(a\vee c, b\vee d), (a\wedge c, b\wedge d)\in\theta_F$. Then θ_F is a lattice congruence on L. Now, we show that $Co\ker\theta_F = F$. Let $y\in Co\ker\theta_F$. Then $(y,1)\in\theta_F$ and hence $y^{\circ\circ}\wedge f^{\circ\circ}\wedge j = 1^{\circ\circ}\wedge f^{\circ\circ}\wedge j$, for some $f\in F$. This gives $y^{\circ\circ}\geq f^{\circ\circ}\wedge j\in F$. Since $y^{\circ\circ}, y\vee j\in F$, then $y = y^{\circ\circ}\wedge(y\vee j)\in F$. Therefore $Co\ker\theta_F\subseteq F$. On the other hand, let $f\in F$. Thus $f\geq j$. Then $f^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j = 1^{\circ\circ}\wedge f_1^{\circ\circ}\wedge j$, for some $f_1\in F$. implies $(f,1)\in\theta_F$. Thus $f\in Co\ker\theta_F$. Then $F\subseteq Co\ker\theta_F$. Consequently, $Co\ker\theta_F = F$ and therefore θ_F is a j-lattice congruence on L.

(2) Let $F\subseteq G$. Suppose that $(x,y)\in\theta_F$. Then $x^{\circ\circ}\wedge f^{\circ\circ}\wedge j = y^{\circ\circ}\wedge f^{\circ\circ}\wedge j$ for some $f\in F\subseteq G$. Thus $(x,y)\in\theta_G$. Therefore $\theta_F\subseteq\theta_G$. Conversely, let $\theta_F\subseteq\theta_G$. Then $F = Co\ker\theta_F\subseteq Co\ker\theta_G = G$. Hence $F\subseteq G$.

Theorem 34. *For any two j-filters F and G of a locally bounded $\underline{K}_2$-algebra L, we have*

(1) $\theta_F\vee\theta_G = \theta_{F\vee G}$,
(2) $\theta_F\cap\theta_G = \theta_{F\cap G}$.

Proof. (1) Since $F, G\subseteq F\vee G$, then by Theorem 33, $\theta_F\subseteq\theta_{F\vee G}$ and $\theta_G\subseteq\theta_{F\vee G}$. Hence $\theta_{F\vee G}$ is an upper bound of θ_F and θ_G. Suppose that θ_H is an upper bound of θ_F and θ_G. Thus we have $\theta_F\subseteq\theta_H$ and $\theta_G\subseteq\theta_H$. Then again by Theorem 33, $F\subseteq H$ and $G\subseteq H$. This gives $F\vee G\subseteq H$ and hence $\theta_{F\vee G}\subseteq\theta_H$. Therefore $\theta_{F\vee G}$ is the least upper bound of θ_F and θ_G. Then $\theta_F\vee\theta_G = \theta_{F\vee G}$.
(2) As $F\cap G\subseteq F$ and G, then by Theorem 33, $\theta_{F\cap G}\subseteq\theta_F$ and $\theta_{F\cap G}\subseteq\theta_G$. Hence $\theta_{F\cap G}$ is a lower bound of θ_F and θ_G. Let θ_H be any lower bound of θ_F and θ_G. Thus we have $\theta_H\subseteq\theta_F$ and $\theta_H\subseteq\theta_G$. Then again by Theorem 33, $H\subseteq F$ and $H\subseteq G$. This implies that $H\subseteq F\cap G$ and $\theta_H\subseteq\theta_{F\cap G}$. Therefore $\theta_{F\cap G}$ is the greatest lower bound of θ_F and θ_G. Then $\theta_F\cap\theta_G = \theta_{F\cap G}$.
Consider $Con_j(L) = \{\theta_F : F\in F_j(L)\}$ as the set of all j-lattice congruences of L that are induced by j-filters of L.

Theorem 35. *Let L be a locally bounded $\underline{K}_2$-algebra. Then* $(Con_j(L);\vee,\wedge,\theta_{L^\vee},\theta_L)$ *is a bounded lattice.*

Proof. Clearly, $\theta_{L^\vee}$ and $\theta_L = \nabla_L$ are the smallest and largest elements of $Con_j(L)$, respectively. Let $\theta_F, \theta_G\in Con_j(L)$ and $F, G\in F_j(L)$. Then by Theorem 34, we have $\theta_F\vee\theta_G = \theta_{F\vee G}$ and $\theta_F\cap\theta_G = \theta_{F\cap G}$.
Thus $(Con_j(L),\vee,\wedge,\theta_{L^\vee},\theta_L)$ is a bounded lattice.

6. Principal j-lattice congruences of a locally bounded $\underline{K}_2$-algebra

In this section, we present and characterize the principal j-lattice congruence on a locally bounded $\underline{K}_2$-algebra L and we study the relationship between the principal lattice congruences and the principal j-lattice congruences on L.

Lemma 36. *If $F = (x)^{\Delta}$, for all $x\in L$, then*

$$(a,b)\in\theta_{(x)^{\Delta}} \Leftrightarrow a^{\circ\circ}\wedge x^{\circ\circ}\wedge j = b^{\circ\circ}\wedge x^{\circ\circ}\wedge j,$$

and $Co\ker\ \theta_{(x)^{\Delta}} = (x)^{\Delta}$.

Proof. *Let* $F = (x)^{\Delta}$. *Let* $(a,b) \in \theta_{(x)^{\Delta}}$. *Then* $a^{\circ\circ} \wedge f^{\circ\circ} \wedge j = b^{\circ\circ} \wedge f^{\circ\circ} \wedge j$, *for some* $f \in (x)^{\Delta} = [x^{\circ\circ} \wedge j)$. *Thus* $f^{\circ\circ} \geq f \geq x^{\circ\circ} \wedge j$. *Then*

$a^{\circ\circ} \wedge f^{\circ\circ} \wedge j \wedge x^{\circ\circ} = b^{\circ\circ} \wedge f^{\circ\circ} \wedge j \wedge x^{\circ\circ}$ implies

$a^{\circ\circ} \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge j$.

Since $(x)^{\Delta}$ is a j-filter of L, then by Theorem 33 (1), $\theta_{(x)^{\Delta}}$ is a j-lattice congruence on L with $Co\ker\theta_{(x)^{\Delta}} = (x)^{\Delta}$.
Now, we show that $\theta_{(x)^{\Delta}}$ is a principal j-lattice congruence on L.

Theorem 37. *Let L be a locally bounded* $\underline{K}_2$*-algebra. Then* $\theta_{(x)^{\Delta}} = \theta_{(x^{\circ\circ} \wedge j, 1)}$, *for all* $x \in L$, *that is,* $\theta_{(x)^{\Delta}}$ *is a principal j-lattice congruence on L.*

Proof. Let $(a,b) \in \theta_{(x^{\circ\circ} \wedge j,1)}$. Then,

$(a,b) \in \theta_{(x^{\circ\circ} \wedge j,1)}$

$\Rightarrow a \wedge x^{\circ\circ} \wedge j = b \wedge x^{\circ\circ} \wedge j$, by Theorem 7

$\Rightarrow a^{\circ\circ} \wedge (a \vee j) \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge (b \vee j) \wedge x^{\circ\circ} \wedge j$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge ((a \vee j) \wedge j) = b^{\circ\circ} \wedge x^{\circ\circ} \wedge ((b \vee j) \wedge j)$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge j$, by the absorption identity,

where $z = z^{\circ\circ} \wedge (z \vee j)$ for all $z \in L$. This gives that $(a,b) \in \theta_{(x)^{\Delta}}$. Hence $\theta_{(x^{\circ\circ} \wedge j,1)} \subseteq \theta_{(x)^{\Delta}}$. Conversely, let $(a,b) \in \theta_{(x)^{\Delta}}$. Then,

$(a,b) \in \theta_{(x)^{\Delta}}$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge j$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge (a \vee j) \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge (b \vee j) \wedge j$,

$\Rightarrow a \wedge x^{\circ\circ} \wedge j = b \wedge x^{\circ\circ} \wedge j$.

Thus $(a,b) \in \theta_{(x^{\circ\circ} \wedge j,1)}$ and hence $\theta_{(x)^{\Delta}} \subseteq \theta_{(x^{\circ\circ} \wedge j,1)}$. Therefore $\theta_{(x)^{\Delta}} = \theta_{(x^{\circ\circ} \wedge j,1)}$.
The following two results are two characterizations of a principal j-lattice congruence of a locally bounded $\underline{K}_2$-algebra L.

Lemma 38. *The principal lattice congruence* $\theta_{(x,1)}$ *of a locally bounded* $\underline{K}_2$*-algebra L is a principal j-lattice congruence on L if and only if* $j \geq x$.

Proof. Let $\theta_{(x,1)}$ be a principal j-lattice congruence on L. Then
$j \in Co\ker\ \theta_{(x,1)}$. Thus $j \wedge x = 1 \wedge x = x$. Hence $j \geq x$. Conversely, let $\theta_{(x,1)}$ be a principal lattice congruence on L and $j \geq x$. Then $j \wedge x = x = 1 \wedge x$ implies $(j,1) \in \theta_{(x,1)}$ and hence $j \in Co\ker\ \theta_{(x,1)}$. Therefore $\theta_{(x,1)}$ is a principal j-lattice congruence on L.

Theorem 39. *Let* $\theta_{(x,1)}$ *be a principal lattice congruence on a locally bounded* $\underline{K}_2$*-algebra L. Then* $\theta_{(x,1)} = \theta_{(x)^{\Delta}}$ *if and only if* $j \geq x$.

Proof. Let $\theta_{(x,1)} = \theta_{(x)^{\Delta}}$. Then $\theta_{(x,1)}$ is a principal j-lattice congruence on L, by Theorem 37. Implies $j \geq x$, by Lemma 38. Conversely, let $j \geq x$. Then by Lemma 38, $\theta_{(x,1)}$ is a principal j-lattice congruence on L. We show that
$\theta_{(x,1)} = \theta_{(x)^{\Delta}}$. Let $(a,b) \in \theta_{(x,1)}$. Then,

$(a,b) \in \theta_{(x,1)}$

$\Rightarrow a \wedge x = b \wedge x$

$\Rightarrow (a \wedge x)^{\circ\circ} = (b \wedge x)^{\circ\circ}$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} = b^{\circ\circ} \wedge x^{\circ\circ}$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge j$.

Hence $(a,b) \in \theta_{(x)^{\Delta}}$ implies $\theta_{(x,1)} \subseteq \theta_{(x)^{\Delta}}$. On the other hand, let $(a,b) \in \theta_{(x)^{\Delta}}$. Since $a = a^{\circ\circ} \wedge (a \vee j)$ and $b = b^{\circ\circ} \wedge (b \vee j)$, we have

$(a,b) \in \theta_{(x)^{\Delta}}$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge j$

$\Rightarrow a^{\circ\circ} \wedge x^{\circ\circ} \wedge (a \vee j) \wedge j = b^{\circ\circ} \wedge x^{\circ\circ} \wedge (b \vee j) \wedge j$,
by the absorption identity

$\Rightarrow a^{\circ\circ} \wedge (a \vee j) \wedge x^{\circ\circ} \wedge j = b^{\circ\circ} \wedge (b \vee j) \wedge x^{\circ\circ} \wedge j$

$\Rightarrow a \wedge x^{\circ\circ} \wedge j = b \wedge x^{\circ\circ} \wedge j$

$\Rightarrow a \wedge x^{\circ\circ} \wedge j \wedge x = b \wedge x^{\circ\circ} \wedge j \wedge x$

$\Rightarrow a \wedge x^{\circ\circ} \wedge x = b \wedge x^{\circ\circ} \wedge x$, as $x \leq j$

$\Rightarrow a \wedge x = b \wedge x$, as $x \leq x^{\circ\circ}$.

Thus $(a,b) \in \theta_{(x,1)}$ and hence $\theta_{(x)^{\Delta}} \subseteq \theta_{(x,1)}$. Therefore $\theta_{(x,1)} = \theta_{(x)^{\Delta}}$.
We denote the set all of principal j-lattice congruences on a locally bounded $\underline{K}_2$-algebra L by $Con_j^p(L) = \{\theta_{(x)^{\Delta}} : (x)^{\Delta} \in F_j^p(L)\}$.

Theorem 40. *Let x and y be any two elements of a locally bounded* $\underline{K}_2$*-algebra L. Then,*

(1) $x \leq y \Leftrightarrow \theta_{(y)^{\Delta}} \subseteq \theta_{(x)^{\Delta}}$.
(2) $\theta_{(x)^{\Delta}} \vee \theta_{(y)^{\Delta}} = \theta_{(x \wedge y)^{\Delta}}$,
(3) $\theta_{(x)^{\Delta}} \cap \theta_{(y)^{\Delta}} = \theta_{(x \vee y)^{\Delta}}$,

(4) $Con_j^p(L)$ is a bounded sublattice of $Con_j(L)$.

Proof. (1) Let $x \leq y$. Then $(y)^\Delta \subseteq (x)^\Delta$, by Theorem 20. This implies that $\theta_{(y)^\Delta} \subseteq \theta_{(x)^\Delta}$, by Theorem 32 (2). Conversely, let $\theta_{(y)^\Delta} \subseteq \theta_{(x)^\Delta}$. Then again, by Theorem 32 (2), $(y)^\Delta \subseteq (x)^\Delta$. This implies that $x \leq y$, by Theorem 20 (4).

(2) For all $\theta_{(x)^\Delta}$ and $\theta_{(y)^\Delta}$, we have

$$\theta_{(x)^\Delta} \vee \theta_{(y)^\Delta} = \theta_{(x)^\Delta \vee (y)^\Delta}, \text{by Theorem 33(1)}$$

$$= \theta_{(x \wedge y)^\Delta}, \text{ by}$$

Theorem 22 (1).

(3) Similarly, we get

$$\theta_{(x)^\Delta} \cap \theta_{(y)^\Delta} = \theta_{(x)^\Delta \cap (y)^\Delta}, \text{by Theorem 33(2)}$$

$$= \theta_{(x \vee y)^\Delta}, \text{ by Theorem 22 (2).}$$

(4) Clearly, $\theta_{(0)^\Delta} = \nabla_L$, $\theta_{(1)^\Delta} = \theta_{L^\vee}$. From (2), (3), we get $(Con_j^p(L); \vee, \wedge, \theta_{(1)^\Delta}, \theta_{(0)^\Delta})$ is a bounded sublattice of $Con_j(L)$.

Theorem 41. *Let L be a locally bounded $\underline{K}_2$-algebra. Then the class $Con_j^p(L)$ of all principal j-lattice congruences forms a Kleene algebra which is isomorphic to $F_j^p(L)$.*

Proof. It is clear that $\theta_{(1)^\Delta} = \theta_{L^\vee}$ and $\theta_{(0)^\Delta} = \nabla_L$ are the least and greatest elements of $Con_j^p(L)$. Now, we prove that $Con_j^p(L)$ is a distributive lattice. Let $\theta_{(x)^\Delta}, \theta_{(y)^\Delta}, \theta_{(z)^\Delta} \in Con_j^p(L)$. Then

$$\theta_{(x)^\Delta} \cap \left[\theta_{(y)^\Delta} \vee \theta_{(z)^\Delta}\right] = \theta_{(x)^\Delta} \cap \theta_{(y \wedge z)^\Delta}$$

$$= \theta_{(x \vee (y \wedge z))^\Delta}, \text{by Theorem 40}$$

$$= \theta_{(x)^\Delta \cap [(y)^\Delta \vee (z)^\Delta]}, \text{by Theorem 22}$$

$$= \theta_{[(x)^\Delta \cap (y)^\Delta] \vee [(x)^\Delta \cap (z)^\Delta]}, \text{by Theorem 26}$$

$$= \theta_{((x \vee y) \cap (x \vee z))^\Delta}, \text{by Theorem 22}$$

$$= \theta_{(x \vee y)^\Delta} \vee \theta_{(x \vee z)^\Delta}, \text{by Theorem 40}$$

$$= \left(\theta_{(x)^\Delta} \cap \theta_{(y)^\Delta}\right) \vee \left(\theta_{(x)^\Delta} \cap \theta_{(z)^\Delta}\right).$$

Hence $Con_j^p(L)$ is a distributive lattice. We now define an operation * on $Con_j^p(L)$ by $\theta^*_{(x)^\Delta} = \theta_{(x^\circ)^\Delta}$, for all $\theta_{(x)^\Delta} \in Con_j^p(L)$. We have

$$\theta^{**}_{(x)^\Delta} = \theta_{(x^{\circ\circ})^\Delta} = \theta_{(x)^\Delta}, \theta^*_{(0)^\Delta} = \theta_{(0^\circ)^\Delta} = \theta_{(1)^\Delta},$$

and

$$\left(\theta_{(x)^\Delta} \cap \theta_{(y)^\Delta}\right)^* = \theta^*_{(x \vee y)^\Delta}, \text{by Theorem 37(2)}$$

$$= \theta_{((x \vee y)^\circ)^\Delta} = \theta_{(x^\circ \wedge y^\circ)^\Delta}$$

$$= \theta_{(x^\circ)^\Delta} \vee \theta_{(y^\circ)^\Delta}, \text{by Theorem 37(2)}$$

$$= \theta^*_{(x)^\Delta} \vee \theta^*_{(y)^\Delta}.$$

Also,

$$\theta_{(x)^\Delta} \cap \theta^*_{(x)^\Delta} = \theta_{(x)^\Delta} \cap \theta_{(x^\circ)^\Delta}$$

$$= \theta_{(x \vee x^\circ)^\Delta}, \text{by Theorem 37(3)}$$

$$= \theta_{(x \vee x^\circ)^\Delta} \subseteq \theta_{(y \wedge y^\circ)^\Delta}, \text{by Theorem 37(1)}$$

$$= \theta_{(y)^\Delta} \vee \theta_{(y^\circ)^\Delta}, \text{by Theorem 37(2)}$$

$$= \theta_{(y)^\Delta} \vee \theta^*_{(y)^\Delta}.$$

Then $(Con_j^p(L); \vee, \cap, ^*, \theta_{L^\vee}, \nabla_L)$ is a Kleene algebra. Now, we show that $Con_j^p(L)$ and $F_j^p(L)$ are isomorphic Kleene algebras. Define a map $f: F_j^p(L) \to Con_j^p(L)$ by $f((a)^\Delta) = \theta_{(a)^\Delta}$, for all $(a)^\Delta \in F_j^p(L)$. Then we have

$$f((a)^\Delta \vee (b)^\Delta) = f((a \wedge b)^\Delta), \text{by Theorem 22(1)}$$

$$= \theta_{(a \wedge b)^\Delta}$$

$$= \theta_{(a)^\Delta} \vee \theta_{(b)^\Delta}, \text{by Theorem 37(2)}$$

$$= f(a) \vee f(b),$$

and

$$f((a)^\Delta \cap (b)^\Delta) = f((a \vee b)^\Delta), \text{by Theorem 22(2)}$$

$$= \theta_{(a \vee b)^\Delta}$$

$$= \theta_{(a)^\Delta} \cap \theta_{(b)^\Delta}, \text{by Theorem 37(3)}$$

$$= f(a) \wedge f(b).$$

Also,

$$\left(f((a)^\Delta)\right)^* = \theta^*_{(a)^\Delta} = \theta_{(a^\circ)^\Delta} = f((a^\circ)^\Delta).$$

Then f is a homomorphism. Let $f((a)^\Delta) = f((b)^\Delta)$. Then $\theta_{(a)^\Delta} = \theta_{(b)^\Delta}$. Implies $a = b$. Hence f is an injective map. Also, for every $\theta_{(a)^\Delta} \in Con_j^P(L)$ we have

$\theta_{(a)^{\Delta}} = f((a)^{\Delta})$, $(a)^{\Delta} \in F_j^p(L)$. Then f is a surjective map. Therefore $F_j^p(L)$ and $Con_j^P(L)$ are isomorphic Kleene algebras.

Example 42. Consider the locally bounded $\underline{K}_2$-algebra L as in Example 27. The principal j-lattice congruences on L are gives as follows:

$\theta_{(1)^{\Delta}} = \theta_{L^{\vee}} = \{\{0\}, \{h,b\}, \{a,f.d\}, L^{\vee}\}$,

$\theta_{(b)^{\Delta}} = \{\{0,a,f,d\}, \{L^{\vee},h,b\}\}$,

$\theta_{(a)^{\Delta}} = \{\{0,h,b\}, \{L^{\vee},a,f,d\}\}$, and $\theta_{(0)^{\Delta}} = \nabla_L$.

The lattice $Con_j^P(L)$ is described as in Fig. 4.
We observe that $(Con_j^P(L),^*)$ is a Kleene algebra, where

$\theta^*_{(1)^{\Delta}} = \theta_{(1^{\circ})^{\Delta}} = \theta_{(0)^{\Delta}}, \theta^*_{(b)^{\Delta}} = \theta_{(b^{\circ})^{\Delta}} = \theta_{(d)^{\Delta}},$
$\theta^*_{(d)^{\Delta}} = \theta_{(d^{\circ})^{\Delta}} = \theta_{(b)^{\Delta}}$, and $\theta^*_{(0)^{\Delta}} = \theta_{(0^{\circ})^{\Delta}} = \theta_{(1)^{\Delta}}$

It is clear that $F_j^p(L)$ and $Con_j^P(L)$ are isomorphic Kleene algebras under the map $(a)^{\Delta} \rightarrow \theta_{(a)^{\Delta}}$.

Definition 43. *Let θ be a lattice congruence on a locally bounded $\underline{K}_2$-algebra L. Then θ is called a congruence on L if*

$(x,y) \in \theta$ implies $(x^{\circ}, y^{\circ}) \in L$.

Now, we give the answer to the following question: whether $\theta_{(x)^{\Delta}}$ is a principal j-congruence on L. To answer this question, we need the following:

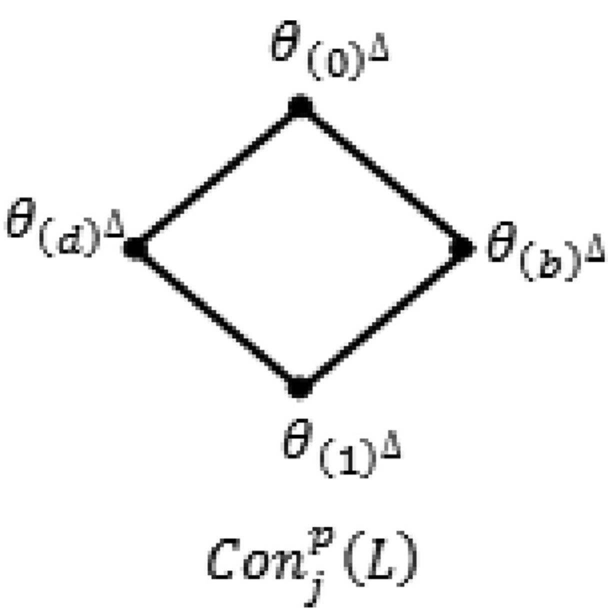


Fig. 4. $Con_j^P(L)$ is a Kleene algebra.

Definition 44. *A Boolean element a is defined as an element of a locally bounded $\underline{K}_2$-algebra L, where $a \vee a^{\circ} = 1$.*

Lemma 45. *The set $B(L) = \{a \in L^{\circ\circ} : a \vee a^{\circ} = 1\}$ is the greatest Boolean subalgebra of $L^{\circ\circ}$.*

Proof. (1) Let $a \in B(L)$. Then we have

$a^{\circ\circ} = 1 \wedge a^{\circ\circ} = (a \vee a^{\circ}) \wedge a^{\circ\circ}$

$= a \vee (a^{\circ} \wedge a^{\circ\circ})$, by modularity of L with $a^{\circ\circ} \geq a$

$= a \vee (a \vee a^{\circ})^{\circ}$

$= a \vee 1^{\circ}$

$= a \vee 0 = a.$

Then $a \in L^{\circ\circ}$ and hence $B(L) \subseteq L^{\circ\circ}$.
It is clear that $0, 1 \in B(L)$. Let $a, b \in B(L)$. Then $a \vee a^{\circ} = 1$ and $b \vee b^{\circ} = 1$. By distributivity of $B(L)$, we get

$(a \wedge b) \vee (a \wedge b)^{\circ} = (a \wedge b) \vee (a^{\circ} \vee b^{\circ})$

$= (a \vee a^{\circ} \vee b^{\circ}) \wedge (b \vee a^{\circ} \vee b^{\circ})$

$= (1 \vee b^{\circ}) \wedge (a^{\circ} \vee 1) = 1.$

Then $a \wedge b \in B(L)$. Similarly, one can show that $a \vee b \in B(L)$. Therefore $B(L)$ is a bounded sublattice of $L^{\circ\circ}$. Let $a \in B(L)$. Then $a^{\circ} \vee a^{\circ\circ} = a^{\circ} \vee a = 1$ and hence $a^{\circ} \in B(L)$. Also, we have $a \wedge a^{\circ} = a^{\circ\circ} \wedge a^{\circ} = (a^{\circ} \vee a)^{\circ} = 1^{\circ} = 0$.
Thus $(B(L); \vee, \wedge, ^{\circ}, 0, 1)$ is a Boolean subalgebra of $L^{\circ\circ}$. Now, we show that $B(L)$ is the greatest Boolean subalgebra of $L^{\circ\circ}$. Consider H be another Boolean subalgebra of $L^{\circ\circ}$. Then for all $a \in H$, we have $a \vee a^{\circ} = 1$ and $a \wedge a^{\circ} = 0$. This implies that $a \in B(L)$ and hence $H \subseteq B(L)$. Therefore $B(L)$ is the greatest Boolean subalgebra of $L^{\circ\circ}$.

Theorem 46. *Let a be a closed element of a locally bounded $\underline{K}_2$-algebra L such that $a \leq j^{\circ\circ}$. Then $\theta_{(a)^{\Delta}}$ is a j-congruence on L if and only if a is a Boolean element of L.*

Proof. Let a be a Boolean element. Then

$a \vee a^{\circ} = 1$ and $a \wedge a^{\circ} = a^{\circ} \wedge a^{\circ\circ} = (a \vee a^{\circ})^{\circ} = 1^{\circ} = 0.$

$\theta_{(a)^{\Delta}}$ is a j-lattice congruence on L, by Theorem 37. Now, we prove that $\theta_{(a)^{\Delta}}$ preserves $^{\circ}$. Let $(x,y) \in \theta_{(a)^{\Delta}}$. Then we get

$(x,y) \in \theta_{(a)^{\Delta}}$

$\Rightarrow x^{\circ\circ} \wedge a^{\circ\circ} \wedge j = y^{\circ\circ} \wedge a^{\circ\circ} \wedge j$

$\Rightarrow (x^{\circ\circ} \wedge a^{\circ\circ} \wedge j)^{\circ} = (y^{\circ\circ} \wedge a^{\circ\circ} \wedge j)^{\circ}$

$\Rightarrow x^{\circ}\vee a^{\circ}\vee j^{\circ}=y^{\circ}\vee a^{\circ}\vee j^{\circ}$, as $z^{\circ\circ\circ}=z^{\circ}$

$\Rightarrow x^{\circ}\vee a^{\circ}=y^{\circ}\vee a^{\circ}$, as $a^{\circ}\geq j^{\circ}$

$\Rightarrow (x^{\circ}\vee a^{\circ})\wedge a^{\circ\circ}=(y^{\circ}\vee a^{\circ})\wedge a^{\circ\circ}$

$\Rightarrow (x^{\circ}\wedge a^{\circ\circ})\vee(a^{\circ}\wedge a^{\circ\circ})=(y^{\circ}\wedge a^{\circ\circ})\vee(a^{\circ}\wedge a^{\circ\circ})$,
by distributivity of $L^{\circ\circ}$

$\Rightarrow x^{\circ}\wedge a^{\circ\circ}=y^{\circ}\wedge a^{\circ\circ}$, as $a^{\circ}\wedge a^{\circ\circ}=0$

$\Rightarrow x^{\circ}\wedge a^{\circ\circ}\wedge j=y^{\circ}\wedge a^{\circ\circ}\wedge j$.

Hence $(x^{\circ},y^{\circ})\in\theta_{(a)^{\Delta}}$. Therefore $\theta_{(a)^{\Delta}}$ is a j-congruence on L. Conversely, let $\theta_{(a)^{\Delta}}$ be a j-congruence on L and $a\leq j^{\circ\circ}$. Then $Co\ker\theta_{(a)^{\Delta}}=(a)^{\Delta}$. Thus $(a,1)\in\theta_{(a)^{\Delta}}$ and hence $(a^{\circ},1^{\circ})\in\theta_{(a)^{\Delta}}$. We get

$(a^{\circ},0)\in\theta_{(a)^{\Delta}}$

$\Rightarrow a^{\circ\circ\circ}\wedge a^{\circ\circ}\wedge j=0^{\circ\circ}\wedge a^{\circ\circ}\wedge j$

$\Rightarrow a^{\circ}\wedge a^{\circ\circ}\wedge j=0$, as $a^{\circ}=a^{\circ\circ\circ}$

$\Rightarrow a^{\circ}\wedge a\wedge j=0$, as $a^{\circ\circ}=a$

$\Rightarrow (a^{\circ}\wedge a\wedge j)^{\circ}=0^{\circ}$

$\Rightarrow a^{\circ\circ}\vee a^{\circ}\vee j^{\circ}=1$

$\Rightarrow a^{\circ\circ}\vee a^{\circ}=1$, as $a^{\circ}\geq j^{\circ}$

$\Rightarrow a\vee a^{\circ}=1$, as $a^{\circ\circ}=a$.

Therefore a is a Boolean element of L.

Example 47. *Consider the locally bounded* $\underline{K}_2$-algebra *L which is represented in Example 27. The set $B(L)=\{0,b,d,1\}$ contains all the closed elements of L.* Now, 0, *b are Boolean elements of L such that $0,b\leq j^{\circ\circ}=c$. So $\theta_{(0)^{\Delta}}=\nabla_L$ and $\theta_{(b)^{\Delta}}=\{\{0,a,f,d\},\{L^{\vee},h,b\}\}$ are* j-congruences on L. *But $d,1\in B(L)$ and $d,1\not\leq j^{\circ\circ}$. So* $\theta_{(d)^{\Delta}}=\{\{0,h,b\},\{L^{\vee},a,f,d\}\}$ and $\theta_{(1)^{\Delta}}=\{\{0\},\{h,b\},\{a,f.d\},L^{\vee}\}$ are not preserve the unary operation $^{\circ}$. Hence $\theta_{(1)^{\Delta}}$ and $\theta_{(d)^{\Delta}}$ are not j-congruences on L.

Ethics information

This paper did not involve any experiments on humans or animals, and therefore no ethical approval was required.

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Authors' contributions

All authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of interest

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