The Bayesian Estimation and Prediction Process Applied to a Mixture of Weibull and Gompertz Distributions Based on Type-I censoring

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The Bayesian Estimation and Prediction Process Applied to a Mixture of Weibull and Gompertz Distributions Based on Type-I Censoring

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Abstract

We examine different methods to estimate the parameters of a lifetime model represented by a mixture of Weibull and Gompertz distributions, based on Type-I censoring. We derive Bayes estimators with a variety of loss functions, including symmetric Squared Error, asymmetric Linear Exponential, and General Entropy, utilizing both informative and noninformative priors. We also go over how to create the model's two-sample Bayesian prediction intervals. To demonstrate these methods, we provide computational results through Monte Carlo simulations and real data.

Keywords: Bayesian estimation and prediction, Gompertz distribution, Loss function, Mixture model, Type-I censoring, Weibull distribution

1. Introduction

Lifetime distributions have recently become of great importance under censoring schemes due to the extensive applications which contribute in different domains. In many life testing and reliability experiments, experimenters are unable to gather complete information or failure times for all experimental units. However, there are several statuses in which items are lost from the test before failure. Gupta [1], compared two types of controlled samples, when the experiment ends after observing a predetermined proportion of observations and also at determined fixed points. The obtained data of such experimenters is known as the censored data (See Balakrishnan *et al.* [2]). The order statistics from a wide range of distributions using Bayesian methods was presented Mohie El-Din *et al.* [3]. A new study in two-sample Bayesian prediction intervals of generalized order statistics based on multiply type- II censored data was introduced by Mohie El-Din *et al.* [4]. Sadek [5] introduced a study on Bayesian prediction utilizing hybrid Type-I censored data drawn from the general distribution class. Mohie El-Din *et al.* [6], studied the statistical inference of the Pareto distribution based on progressive type-I hybrid censoring scheme. These censoring schemes include some obstacles, as these only allow to removal units at the ultimate values of experiment (See Soliman [7]).

A failure population was split into two subpopulations by Mendenhall and Hader [8], each of which represented a distinct kind of failure. Chen *et al.* [9], extended the idea of Berkson and Gage [10], to become a two-model mixture model for analyzing cancer survival data. Gordon [11,12] proposed that a mixture of two subpopulations may be utilized to represent the survival function of cancer patients receiving treatment. He modeled the survival time distribution of both subpopulations using the Gompertz distribution. Masuyama [13] utilized a mixture of two gamma distributions for rheumatoid arthritis. Radhakrishna *et al.* [14], derived both moment and maximum likelihood estimators for the unknown parameters of a two-component mixture

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of the generalized gamma distribution. Ahmed *et al.* [15] obtained approximate Bayes estimators for the parameters of a mixture of two Weibull distributions under type-II censoring. Jaheen [16], derived prediction bounds for the s^{th} future observation under a mixture of a two-component Gompertz lifetime model. Also, Jaheen [17] addressed the issue of estimating the parameters of a finite mixture of two exponential distributions using record statistics. Erisoglu *et al.* [18], suggested another mixture that consists of two different distributions to model heterogeneous survival data. Mixture of Burr type XII distribution was offered by Ahmed *et al.* [19]. This paper aims to utilize Bayesian methodology to estimate parameters and derive two-sample prediction bounds for future observations based on the proposed model, considering Type-I censoring. The subsequent sections are structured as follows: Section 2 introduces the population and the model. Section 3 delves into Bayesian estimation. Section 4 presents Bayesian two-sample prediction. Section 5 conducts a Monte Carlo simulation study to compare the performance of various parameter estimation methods and presents real data analysis. Finally, conclusions are summarized in Section 6.

2. The population and the model

In this section, we present the population and the model for a mixture of distribution under type-I censoring. A random variable X is said to have a mixture of two component Weibull and Gompertz distribution if its probability density function (pdf) can be written

$$f(x) = \sum_{j=1}^{2} p_j f_j(x), \tag{1}$$

where

$$f_1(x) = \alpha_1 \theta\, x^{\theta-1}\, e^{-\alpha_1 x^\theta}, x>0, \alpha_1>0, \theta>0,$$

$$f_2(x) = \alpha_2\, e^{x-\alpha_2(e^x-1)}, x>0, \alpha_2>0\,.$$

The mixing proportions P_j are such that $0 \le p_j \le 1$, $\sum_{j=1}^{2} p_j = 1$.

The (cdf) is given by:

$$F(x) = \sum_{j=1}^{2} p_j F_j(x), \tag{2}$$

where

$$F_1(x) = 1 - e^{-\alpha_1 x^\theta}, F_2(x) = 1 - e^{-\alpha_2(e^x-1)}.$$

The reliability function can be written

$$R(x) = \sum_{j=1}^{2} p_j R_j(x), \tag{3}$$

$$R_1(x) = e^{-\alpha_1 x^\theta}, R_2(x) = e^{-\alpha_2(e^x-1)}\,.$$

The likelihood function based on type-I censored can be written

$$L(\alpha_1, \alpha_2, \theta, p\,|\underline{x}) = \frac{n!}{(n-r)!}\left[\prod_{j=1}^{r_1} p_1 f_1(x_{1j})\right] \left[\prod_{j=1}^{r_2} p_2 f_2(x_{2j})\right][1-F(t)]^{n-r}, \tag{4}$$

where

$$f_1(x_{1j}) = \alpha_1\, \theta\, x_{1j}^{\theta-1}\, e^{-\alpha_1 x_{1j}^\theta}, x>0, \alpha_1>0, \theta>0,$$

$$f_2(x_{2j}) = \alpha_2\, e^{x_{2j}-\alpha_2\left(e^{x_{2j}}-1\right)}, x>0, \alpha_2>0.$$

Then,

$$L(\alpha_1, \alpha_2, \theta, p|\underline{x}) \propto \prod_{j=1}^{r_1} p_1 \alpha_1 \theta x_{1j}^{\theta-1}\, e^{-\alpha_1 x_{1j}^\theta} \prod_{j=1}^{r_2} p_2 \alpha_2 e^{x_{2j}-\alpha_2\left(e^{x_{2j}}-1\right)} [R(t)]^{n-r}. \tag{5}$$

Put $p_1 = p$ and $p_2 = 1-p, r = r_1 + r_2$ and $R(t) = \sum_{j=1}^{2} p_j R_j(x)$.Assuming that, the parameter θ is known, the likelihood function (5) reduces to

$$L(\alpha_1, \alpha_2, p|\underline{x}) \propto \sum_{k=0}^{n-r} \binom{n-r}{k} p^{n-k-r_2}(1-p)^{r_2+k} \alpha_1^{r_1}\alpha_2^{r_2} \times e^{-\alpha_1\left(\sum_{j=1}^{r_1} x_{1j}^\theta + (n-r-k)\, t^\theta\right)} e^{-\alpha_2\left(\sum_{j=1}^{r_2}\left(e^{x_{2j}}-1\right)+k\left(e^t-1\right)\right)}. \tag{6}$$

3. Bayesian estimation

In this section, we derive Bayesian estimators of the parameters α_1, α_2 and p of the considered model under Type-I censoring. Suppose the prior distribution of the parameters α_1, α_2 and p are $\alpha_1 \sim$ Gamma (a_1, b_1), $\alpha_2 \sim$ Gamma (a_2, b_2), and $p \sim$ Beta (c, d). Suppose, now, the independence of parameters, the joint distribution prior for α_1, α_2 and p is:

$$\pi(\alpha_1, \alpha_2, p) = \pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(p), \tag{7}$$

where

$$\pi_i(\alpha_i) = \frac{b_i^{a_i}}{\Gamma(a_i)}\, \alpha_i^{a_i-1}\, e^{-b_i\alpha_i}, \alpha_i > 0, a_i, b_i > 0; i = 1, 2$$

$$\pi_3(p) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\, p^{c-1}\,(1-p)^{d-1}, 0 < p < 1, c, d > 0. \quad (8)$$

Then, the joint prior distribution of α_1, α_2 and p are given

$$\pi\,(\alpha_1, \alpha_2, p) \propto \left[\prod_{i=1}^{2} \alpha_i^{a_i-1}\, e^{-b_i\alpha_i}\right] \times p^{\,c-1}\,(1-p)^{\,d-1}. \quad (9)$$

From Eqs (6) and (9), that the joint posterior density function of α_1, α_2 and p is given by

$$g\,(\alpha_1, \alpha_2, p|\underline{x}) = k_1^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} p^{n-k-r_2+c-1} (1-p)^{r_2+k+d-1}\, \alpha_1^{r_1+a_1-1} \alpha_2^{r_2+a_2-1} e^{-\alpha_1\phi_{1k}}\, e^{-\alpha_2\phi_{2k}}, \quad (10)$$

where,

$$k_1 = \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1, \psi_2)\, \frac{\Gamma(r_1+a_1)\,\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1}\,(\phi_{2k})^{r_2+a_2}},$$

and

$$\phi_{1k} = b_1 + \left(\sum_{j=1}^{r_1} x_{1j}^{\theta} + (n-r-k)\, t^{\,\theta}\right),$$

$$\phi_{2k} = b_2 + \left(\sum_{j=1}^{r_2} (e^{x_{2j}} - 1) + k\,(e^t - 1)\right),$$

$$\psi_1 = n - k - r_2 + c, \psi_2 = r_2 + k + d,$$

where a_1, a_2, b_1, b_2, c and d are the hyper-parameters. Particularity if $a_1 = a_2 = b_1 = b_2 = 0$ and c = d = 1, the case of non-informative improper prior, and are given by

$$\pi_i(\alpha_i) \propto \frac{1}{\alpha_i}\ \alpha_i > 0, i = 1, 2,$$

$$\pi_3(p) = 1, 0 < p < 1. \quad (11)$$

Then , the joint prior distribution of α_1, α_2 and p can be written as follows

$$\pi\,(\alpha_1, \alpha_2, p) \propto \frac{1}{\alpha_1\alpha_2}, \alpha_1, \alpha_2 > 0, 0 < p < 1. \quad (12)$$

From Eq (6) and (12), that, the joint posterior density function of α_1, α_2 and p is given by

$$g\,(\alpha_1, \alpha_2, p|\underline{x}) \propto k_2^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} p^{n-k-r_2} (1-p)^{r_2+k}\, \alpha_1^{r_1-1}\, \alpha_2^{r_2-1} e^{-\alpha_1\phi_{1k}^*}\, e^{-\alpha_2\phi_{2k}^*}, \quad (13)$$

where

$$k_2 = \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1^*, \psi_2^*)\, \frac{\Gamma(r_1)\,\Gamma(r_2)}{(\phi_{1k}^*)^{r_1}\,(\phi_{2k}^*)^{r_2}},$$

and

$$\phi_{1k}^* = \left(\sum_{j=1}^{r_1} x_{1j}^{\theta} + (n-r-k)\, t^{\,\theta}\right),$$

$$\phi_{2k}^* = \left(\sum_{j=1}^{r_2} (e^{x_{2j}} - 1) + k\,(e^t - 1)\right),$$

$$\psi_1^* = n - k - r_2 + 1, \psi_2^* = r_2 + k + 1.$$

3.1. Bayes estimator based on squared error loss function (SE)

The Bayes estimators of α_1, α_2 and p based on the squared error loss function are given by:

$$\widehat{\alpha}_{1_{SE}} = k_1^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1, \psi_2)\, \frac{\Gamma(r_1+a_1+1)\,\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1+1}\,(\phi_{2k})^{r_2+a_2}},$$

$$\widehat{\alpha}_{1_{SEN}} = k_2^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1^*, \psi_2^*)\, \frac{\Gamma(r_1+1)\,\Gamma(r_2)}{(\phi_{1k}^*)^{r_1+1}\,(\phi_{2k}^*)^{r_2}},$$

$$\widehat{\alpha}_{2_{SE}} = k_1^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1, \psi_2)\, \frac{\Gamma(r_1+a_1)\,\Gamma(r_2+a_2+1)}{(\phi_{1k})^{r_1+a_1}\,(\phi_{2k})^{r_2+a_2+1}},$$

$$\widehat{\alpha}_{2_{SEN}} = k_2^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1^*, \psi_2^*)\, \frac{\Gamma(r_1)\,\Gamma(r_2+1)}{(\phi_{1k}^*)^{r_1}\,(\phi_{2k}^*)^{r_2+1}},$$

$$\widehat{p}_{SE} = k_1^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1+1, \psi_2)\, \frac{\Gamma(r_1+a_1)\,\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1}\,(\phi_{2k})^{r_2+a_2}},$$

$$\widehat{p}_{SEN} = k_2^{-1} \sum_{k=0}^{n-r} \binom{n-r}{k} \beta(\psi_1^*+1, \psi_2^*)\, \frac{\Gamma(r_1)\,\Gamma(r_2)}{(\phi_{1k}^*)^{r_1}\,(\phi_{2k}^*)^{r_2}}.$$

3.2. Bayes estimator based on linear exponential loss function (LINEX)

The Bayes estimators of α_1, α_2, and p based on the linear exponential loss function are given by:

$$\widehat{\alpha}_{1_{LINEX}} = -\frac{1}{q}\ln\left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1,\psi_2)\frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(\phi_{1k}+q)^{r_1+a_1}(\phi_{2k})^{r_2+a_2}}\right],$$

$$\widehat{\alpha}_{1_{LINEXN}} = -\frac{1}{q}\ln\left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1^*,\psi_2^*)\frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_{2k}^*)^{r_2}}\right],$$

$$\widehat{\alpha}_{2_{LINEX}} = -\frac{1}{q}\ln\left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1,\psi_2)\frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1}(\phi_{2k}+q)^{r_2+a_2}}\right],$$

$$\widehat{\alpha}_{2_{LINEXN}} = -\frac{1}{q}\ln\left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1^*,\psi_2^*)\frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_{1k}^*)^{r_1}(\phi_{2k}^*+q)^{r_2}}\right],$$

$$\widehat{p}_{LINEX} = -\frac{1}{q}\ln\left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1}(\phi_{2k})^{r_2+a_2}}\sum_{v=0}^{\infty}\frac{(-q)^v}{v!}\beta(\psi_1+v,\psi_2)\right],$$

$$\widehat{p}_{LINEXN} = -\frac{1}{q}\ln\left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_{1k}^*)^{r_1}(\phi_{2k}^*)^{r_2}}\sum_{v=0}^{\infty}\frac{(-q)^v}{v!}\beta(\psi_1^*+v,\psi_2^*)\right].$$

3.3. Bayes estimator based on general entropy loss function (GE)

The Bayes estimators of α_1, α_2 and p based on general entropy loss function are given by:

$$\widehat{\alpha}_{1,GE} = \left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1,\psi_2)\frac{\Gamma(r_1+a_1-h)\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1-h}(\phi_{2k})^{r_2+a_2}}\right]^{-\frac{1}{h}},$$

$$\widehat{\alpha}_{1_{GEN}} = \left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1^*,\psi_2^*)\frac{\Gamma(r_1-h)\Gamma(r_2)}{(\phi_{1k}^*)^{r_1-h}(\phi_{2k}^*)^{r_2}}\right]^{-\frac{1}{h}},$$

$$\widehat{\alpha}_{2,GE} = \left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1,\psi_2)\frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2-h)}{(\phi_{1k})^{r_1+a_1}(\phi_{2k})^{r_2+a_2-h}}\right]^{-\frac{1}{h}},$$

$$\widehat{\alpha}_{2_{GEN}} = \left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1^*,\psi_2^*)\frac{\Gamma(r_1)\Gamma(r_2-h)}{(\phi_{1k}^*)^{r_1}(\phi_{2k}^*)^{r_2-h}}\right]^{-\frac{1}{h}},$$

$$\widehat{p}_{GE} = \left[k_1^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1-h,\psi_2)\frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{(\phi_{1k})^{r_1+a_1}(\phi_{2k})^{r_2+a_2}}\right]^{-\frac{1}{h}},$$

$$\widehat{p}_{GEN} = \left[k_2^{-1}\sum_{k=0}^{n-r}\binom{n-r}{k}\beta(\psi_1^*-h,\psi_2^*)\frac{\Gamma(r_1)\Gamma(r_2)}{(\phi_{1k}^*)^{r_1}(\phi_{2k}^*)^{r_2}}\right]^{-\frac{1}{h}}.$$

4. Bayesian two-sample prediction

In this part we discuss the Bayesian prediction of future order statistics. Utilizing observed Type-I censored. A future sample of size m is randomly selected, independent of another sample of size n, from the identical population as described in Eq (1). Consequently Y_s denotes the s^{th} ordered statistic in the future sample of size m where, $1 \leqslant s \leqslant m$. The s^{th} order statistic in a sample of size m represents the life length of a (m − s+1) out of m system. The distribution function of Y_s, the ordered future sample is given by (See, Arnold *et al.* [20] and Jaheen [16]),

$$\left[F_X\left(y_s\middle|\alpha_1,\alpha_2,p\right)\right]^l\left[1-F_X\left(y_s\middle|\alpha_1,\alpha_2,p\right)\right]^{m-l}$$

$$=\sum_{l=s}^{m}\sum_{j_1=0}^{l}\binom{m}{l}\binom{l}{j_1}(-1)^{j_1}[R(ys)]^{m-l+j_1}, \tag{14}$$

where $F_X(y_s|\alpha_1,\alpha_2,p)=1-R(y_s)$ is the distribution function of the mixture model and $R(y_s)$ is the reliability function of the mixture model after replacing x by y_s. Using the binomial expansion for $[R(y_s)]^{m-l+j_1}$ as follows:

$$\left[R(y_s)\right]^{m-l+j_1}=\left[p_1\,e^{-\alpha_1 y_s^{\theta}}+p_2\,e^{-\alpha_2\left(e^{y_s}-1\right)}\right]^{m-l+j_1}$$

$$=\sum_{j_2=0}^{m-l+j_1}\binom{m-l+j_1}{j_2}p_1^{\delta_1}\,p_2^{j_2}\,e^{-\alpha_1\delta_1 y_s^{\theta}}\,e^{-\alpha_2 j_2\left(e^{y_s}-1\right)}. \tag{15}$$

Therefor, we get

$$F_{Y_s}\left(y_s\middle|\alpha_1,\alpha_2,p\right)=\sum_{l=s}^{m}\sum_{j_1=0}^{l}\sum_{j_2}^{m-l+j_1}\binom{m}{l}\binom{l}{j_1}$$
$$\times\binom{m-l+j_1}{j_2}(-1)^{j_1}\times p_1^{\delta_1}\,p_2^{j_2}\left(e^{-\alpha_1 y_s^{\theta}}\right)^{\delta_1}\left(e^{-\alpha_2\left(e^{y_s}-1\right)}\right)^{j_2}, \tag{16}$$

where $\delta_1=m-l+j_1-j_2$.

The Bayes predictive probability density function of the component in a future sample given $\underline{x}$ is defined by

$$f^*\left(y_s\middle|\underline{x}\right)=\int_0^1\int_0^\infty\int_0^\infty f\left(y_s\middle|\alpha_1,\alpha_2,p\right)g\left(\alpha_1,\alpha_2,p\middle|\underline{x}\right)d\alpha_1 d\alpha_2 dp, \tag{17}$$

where $f(y_s|\alpha_1,\alpha_2,p)$ is the pdf of s^{th} component in a future sample, and $g(\alpha_1,\alpha_2,p|\underline{x})$ the joint posterior

Table 1. Average estimates and corresponding MSE of the parameter $\alpha_1 = 0.2$ based on informative and non-informative prior.

Bayes		SE		LINEX				GE			
				Informative		Noninformative		Informative		Noninformative	
n	T	Info	Noninfo	q = −0.5	q = 0.5	q = −0.5	q = 0.5	h = −0.5	h = 0.5	h = −0.5	h = 0.5
20	2	0.1002 (0.01491)	0.0952 (0.01527)	0.1011 (0.0128)	0.0993 (0.0127)	0.0943 (0.0139)	0.0961 (0.0138)	0.0922 (0.0145)	0.0757 (0.0131)	0.0869 (0.01542)	0.0696 (0.0141)
	3	0.1229 (0.00872)	0.1224 (0.00862)	0.1236 (0.0085)	0.1221 (0.0084)	0.1235 (0.0088)	0.122 (0.0087)	0.1172 (0.0094)	0.1056 (0.0013)	0.1171 (0.00994)	0.1054 (0.0099)
40	2	0.1008 (0.01081)	0.0969 (0.01092)	0.1023 (0.0108)	0.1004 (0.0107)	0.0972 (0.0118)	0.0972 (0.0117)	0.0965 (0.0115)	0.0973 (0.0112)	0.0903 (0.01239)	0.0934 (0.1139)
	3	0.1168 (0.00752)	0.11405 (0.00717)	0.1171 (0.0079)	0.1165 (0.0078)	0.1143 (0.0085)	0.1137 (0.0084)	0.1143 (0.0083)	0.1091 (0.0082)	0.11140 (0.00885)	0.1062 (0.0876)
60	2	0.0971 (0.01038)	0.0944 (0.01152)	0.0974 (0.0105)	0.0969 (0.0103)	0.0946 (0.0116)	0.0942 (0.0115)	0.0947 (0.0114)	0.0898 (0.0112)	0.0924 (0.01226)	0.0874 (0.0118)
	3	0.113 (0.00713)	0.111 (0.00832)	0.1132 (0.0084)	0.1128 (0.0083)	0.1112 (0.0082)	0.1108 (0.0081)	0.1112 (0.0082)	0.1074 (0.0081)	0.1091 (0.00873)	0.1054 (0.0085)

Table 2. Average estimates and corresponding MSE of the parameter $\alpha_2 = 0.1$ based on informative and noninformative prior.

Bayes											
				LINEX				GE			
		SE		Informative		Noninformative		Informative		Noninformative	
n	T	Info	Noninfo	$q = -0.5$	$q = 0.5$	$q = -0.5$	$q = 0.5$	$h = -0.5$	$h = 0.5$	$h = -0.5$	$h = 0.5$
20	c	0.1051 (0.00222)	0.1052 (0.00232)	0.1059 (0.0023)	0.1043 (0.00218)	0.1059 (0.00232)	0.1045 (0.00221)	0.0993 (0.09932)	0.1015 (0.09057)	0.0995 (0.00211)	0.0877 (0.00205)
	3	0.0737 (0.00143)	0.0727 (0.00145)	0.0734 (0.0015)	0.0726 (0.00145)	0.0731 (0.00146)	0.0723 (0.00145)	0.0673 (0.00168)	0.0573 (0.00135)	0.0672 (0.00171)	0.0571 (0.00139)
40	2	0.1026 (0.00142)	0.1026 (0.00146)	0.1031 (0.0014)	0.1021 (0.00138)	0.103 (0.00145)	0.1021 (0.00142)	0.0986 (0.00135)	0.0903 (0.00134)	0.0983 (0.00136)	0.0902 (0.00135)
	3	0.0797 (0.00122)	0.0796 (0.00124)	0.0823 (0.0012)	0.0794 (0.00102)	0.0799 (0.00123)	0.0792 (0.00121)	0.07601 (0.00132)	0.0683 (0.00126)	0.0751 (0.00133)	0.0682 (0.00128)
60	2	0.1035 (0.00092)	0.1035 (0.00093)	0.1038 (0.0009)	0.1033 (0.00093)	0.1037 (0.00096)	0.1032 (0.00094)	0.1012 (0.00091)	0.0965 (0.00087)	0.1012 (0.000912)	0.0963 (0.000878)
	3	0.0934 (0.00063)	0.0934 (0.00064)	0.0936 (0.0007)	0.0932 (0.00067)	0.0936 (0.00068)	0.0933 (0.00066)	0.0913 (0.00065)	0.0879 (0.00064)	0.0919 (0.00096)	0.0876 0.00061

Table 3. Average estimates and corresponding MSE of the parameter $p = 0.7$ based on informative and noninformative prior.

Bayes											
				LINEX				GE			
		SE		Informative		Noninformative		Informative		Noninformative	
n	T	Info	Noninfo	$q = -0.5$	$q = 0.5$	$q = -0.5$	$q = 0.5$	$h = -0.5$	$h = 0.5$	$h = -0.5$	$h = 0.5$
20	2	0.6831 (0.00347)	0.6812 (0.00453)	0.6835 (0.00351)	0.6862 (0.00031)	0.6734 (0.0054)	0.6798 (0.00241)	0.6815 (0.00041)	0.6826 (0.00037)	0.6772 (0.00981)	0.6792 (0.00881)
	3	0.6948 (0.00051)	0.6894 (0.00072)	0.6952 (0.00051)	0.6965 (0.00045)	0.68123 (0.00234)	0.68523 (0.00373)	0.6933 (0.00031)	0.6943 (0.00039)	0.68134 (0.00543)	0.68223 (0.00432)
40	2	0.6849 (0.00341)	0.6811 (0.00512)	0.6844 (0.00042)	0.6837 (0.00035)	0.67751 (0.00734)	0.68051 (0.00673)	0.6825 (0.00033)	0.6835 (0.00036)	0.67098 (0.00851)	0.6889 (0.00581)
	3	0.6986 (0.00043)	0.6923 (0.00092)	0.6929 (0.00041)	0.6983 (0.00033)	0.6826 (0.00861)	0.68664 (0.00655)	0.6971 (0.00032)	0.6981 (0.00035)	0.68332 (0.000664)	0.6895 (0.00089)
60	2	0.6981 (0.00114)	0.6825 (0.00314)	0.6974 (0.00023)	0.6986 (0.0001)	0.6882 (0.0065)	0.6899 (0.00523)	0.6975 (0.00089)	0.69721 (0.00058)	0.68843 (0.000941)	0.68771 (0.000755)
	3	0.6963 (0.00042)	0.6833 (0.000443)	0.6984 (0.00013)	0.6988 (0.00012)	0.6889 (0.00045)	0.67977 (0.00034)	0.6989 (0.000212)	0.69965 (0.00025)	0.69122 (0.000854)	0.92113 (0.000513)

density for α_1, α_2 and p. Therefore, Bayesian prediction density of Y_s for a given value v , is given $P[Y_s \geq v|\underline{x}] = \int_v^\infty f^*(y_s|\underline{x})dy_s$

$$= 1 - \int_0^1 \int_0^\infty \int_0^\infty F_{Y_s}(v|\alpha_1, \alpha_2, p) g(\alpha_1, \alpha_2, p|\underline{x}) d\alpha_1 d\alpha_2 dp, \tag{18}$$

Substitution of (10) and (16) in (18) we get

$$P[Y_s \geq v\,|\underline{x}] = 1 - k_1^{-1} \sum B\,\beta(\psi_1^{**}, \psi_2^{**})\,\frac{\Gamma(r_1 + a_1)\,\Gamma(r_2 + a_2)}{(\phi_{1k}^{**})^{r_1 + a_1}\,(\phi_{2k}^{**})^{r_2 + a_2}},$$

where

$$\sum = \sum_{k=0}^{n-r} \sum_{l=s}^{m} \sum_{j_1=0}^{l} \sum_{j_2}^{m-l+j_1},$$

$$B = \binom{n-r}{k}\binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1},$$

$$\psi_1^{**} = \psi_1 + \delta_1, \psi_2^{**} = \psi_2 + j_2, \phi_{1k}^{**} = \delta_1 v^\theta + \phi_{1k}, \phi_{2k}^{**} = j_2 (e^v - 1) + \phi_{2k}.$$

Substitution of (13) and (16) in (18), we get

$$p[Y_s \geq v\,|\underline{x}] = 1 - k_2^{-1} \sum B\ \beta(\psi_1^{***}, \psi_2^{***})\,\frac{\Gamma(r_1)\,\Gamma(r_2)}{(\phi_{1k}^{***})^{r_1}\,(\phi_{2k}^{***})^{r_2}},$$

where

$$\phi_{1k}^{***} = \delta_1 v^\theta + \phi_{1k}^*, \phi_{2k}^{***} = j_2(e^v - 1) + \phi_{2k}^*, \psi_1^{***} = \psi_1^* + \delta_1, \psi_2^{**} = \psi_2^* + j.$$

A 100 γ % prediction interval for Y_s is the given by

$$P\,[L(\underline{x}) < Y_s < U(\underline{x})] = \gamma,$$

where the following two equations are solved to give L(x) and U(x), respectively:

$$P[Y_s > L(\underline{x})] = \frac{1+\gamma}{2} \text{ and } P[Y_s > U(\underline{x})] = \frac{1-\gamma}{2}. \tag{19}$$

5. Simulation study and numerical example

In this section we give some simulation results in this part to compare the performance of various estimates on Type-I censored basis. The subsequent actions were taken into consideration:

(a) For the parameters, we have considered $(\alpha_1, \alpha_2, p) = (0.2, 0.1, 0.7)$ with $\theta = 2$. The values of q and h are $(0.5, -0.5)$ for (LINEX) and (GE) loss functions. The method of choosing the hyperparameters values introduced by Ahmadi *et al.* [21], ($a_1 = 0.03, a_2 = 0.04, b_1 = 0.2, b_2 = 0.35$, $c = 96.231$, and $d = 41.167$) for informative prior. In case of noninformative prior, we take {($a_1 = a_2 = b_1 = b_2 = 0$), ($c = d = 1$)}. In all these cases, we take the samples of size $n = 20, 40$, and 60 are generated.

Table 4. The informative prior for Ys based on the 95% Bayesian prediction boundaries, lengths, and their simulated coverage probability.

(n, m)	Y1			Y4			Ym		
	(L, U)	Length	Percent	(L, U)	Length	Percent	(L, U)	Length	Percent
(15, 8)	(0.06845, 1.8137)	1.77531	0.819	(0.9449, 3.3409)	2.4960	0.986	(2.3768, 20.5931)	18.2163	0.919
(20, 8)	(0.06797, 1.8335)	1.7655	0.839	(0.90448, 3.348)	2.4944	0.987	(2.4315, 25.0693)	22.6378	0.895
(30, 8)	(0.08008, 1.8265)	1.7464	0.826	(1.0038, 3.1879)	2.4841	0.987	(2.5026, 28.778)	26.2754	0.875
(50, 8)	(0.09002, 1.8204)	1.7303	0.812	(1.0588, 3.1035)	2.4747	0.979	(2.5293, 30.1986)	27.6693	0.878
	Y1			Y5			Ym		
(15, 10)	(0.05692, 1.6208)	1.5938	0.808	(1.00911, 3.3157)	2.3066	0.998	(2.5802, 24.8702)	22.291	0.910
(20, 10)	(0.06109, 1.6409)	1.5798	0.766	(1.07743, 3.1766)	2.0992	0.987	(2.6143, 26.816)	24.2016	0.866
(30, 10)	(0.07092, 1.6380)	1.5471	0.81	(1.13391, 3.0808)	1.9469	0.987	(2.6852, 30.5792)	27.894	0.823
(50, 10)	(0.06091, 1.6626)	1.5017	0.782	(1.16741, 3.0432)	1.8762	0.973	(2.7482, 31.8034)	29.0552	0.768
	Y1			Y6			Ym		
(15, 12)	(0.04897, 1.4660)	1.4970	0.751	(1.07488, 3.2454)	2.1705	0.995	(2.6985, 25.1321)	22.4335	0.852
(20, 12)	(0.04470, 1.5139)	1.4692	0.741	(1.1817, 3.0946)	1.9129	0.988	(2.7585, 29.863)	27.1045	0.848
(30, 12)	(0.06674, 1.4705)	1.4438	0.751	(1.2264, 3.0440)	1.8175	0.981	(2.8291, 31.4186)	28.5894	0.778
(50, 12)	(0.06534, 1.4714)	1.4060	0.742	(1.2507, 2.9858)	1.7350	0.979	(2.9007, 31.9038)	29.003	0.719
	Y1			Y7			Ym		
(15, 13)	(0.03935, 1.4453)	1.4060	0.723	(1.1436, 3.4789)	2.3353	0.996	(2.7522, 25.6551)	22.9027	0.833
(20, 13)	(0.04962, 1.4242)	1.3746	0.736	(1.2831, 3.2724)	1.9892	0.978	(2.8262, 29.9419)	27.1156	0.813
(30, 13)	(0.05288, 1.4396)	1.3567	0.734	(1.3217, 3.2200)	1.8982	0.985	(2.8934, 31.6058)	28.7123	0.736
(50, 13)	(0.06292, 1.4153)	1.3224	0.729	(1.4054, 3.0764)	1.6710	0.968	(2.9655, 31.9806)	29.0151	0.684

(b) Generate a random number u uniformly distributed in the interval (0, 1), if $u \leq p$, the observation has been randomly selected from the first subpopulation; otherwise, if $u > p$, the observation has been selected from the second subpopulation.
(c) Calculate the Bayes estimates of proposed parameter under informative and non-informative prior based on different loss function.
(d) To derive Bayesian prediction intervals for future observations of Y_s equation (19) are solved numerically with $\gamma = 0.95$
(e) The described steps are iterated 1000 times, and the averages of the estimates along with the mean squared errors are computed and displayed in (Tables 1–3).
(f) The average lower and upper intervals of Y_s are calculated when $s = \frac{m}{2}$ for even m or $\frac{m+1}{2}$ for odd m, across various sample sizes n and different sizes of future samples m. Simulated coverage probabilities and average interval lengths are presented in (Tables 4 and 5). The results were obtained by using Mathematica 10.

Table 5. The simulated coverage probability, lengths, and 95% Bayesian prediction ranges for Ys-based noninformative prior.

(n, m)	Y1			Y4			Ym		
	(L, U)	Length	Percent	(L, U)	Length	Percent	(L, U)	Length	Percent
(15, 8)	(0.07412, 1.9199)	1.8458	0.819	(0.99952, 3.4403)	2.4407	0.984	(2.36872, 18.7678)	16.3991	0.980
(20, 8)	(0.08026, 1.8827)	1.8024	0.832	(1.0262, 3.3528)	2.3266	0.981	(2.50903, 25.9091)	23.4001	0.854
(30, 8)	(0.09623, 1.8644)	1.7680	0.826	(1.04227, 2.21552)	2.2286	0.975	(2.54061, 28.9641)	26.4235	0.839
(50, 8)	(0.09747, 1.8375)	1.7400	0.832	(1.06433, 3.1812)	2.1169	0.972	(2.57912, 30.6502)	28.071	0.846
	Y1			Y5			Ym		
(15, 10)	(0.06408, 1.7383)	1.6742	0.788	(1.0998, 3.3675)	2.2676	0.986	(2.64996, 18.2361)	15.5861	0.851
(20, 10)	(0.06375, 1.6973)	1.6335	0.792	(1.1239, 3.25392)	2.1301	0.989	(2.69597, 27.932)	25.2361	0.808
(30, 10)	(0.06545, 1.6788)	1.6134	0.782	(1.1640, 3.1898)	2.0258	0.979	(2.72633, 30.8963)	28.17	0.781
(50, 10)	(0.07637, 1.6540)	1.57760	0.790	(1.17485, 3.0960)	1.9212	0.974	(2.77358, 31.8209)	29.0473	0.744
	Y1			Y6			Ym		
(15, 12)	(0.05413, 1.5713)	1.5172	0.733	(1.18049, 3.3389)	2.1584	0.984	(2.79412, 20.4786)	17.6844	0.824
(20, 12)	(0.05885, 1.5505)	1.4917	0.772	(1.19667, 3.2539)	2.0572	0.986	(2.83608, 27.5601)	24.724	0.777
(30, 12)	(0.06097, 1.5125)	1.4516	0.731	(1.22489, 3.1200)	1.8951	0.978	(2.88359, 31.5684)	28.6848	0.742
(50, 12)	(0.06603, 1.5003)	1.4342	0.732	(1.25311, 3.0518)	1.7987	0.979	(2.91859, 31.9718)	29.0532	0.698
	Y1			Y7			Ym		
(15, 13)	(0.05560, 1.5087)	1.4531	0.763	(1.29163, 3.5190)	2.2274	0.988	(2.84397, 22.6998)	19.8559	0.769
(20, 13)	(0.06328, 1.4708)	1.4075	0.725	(1.32725, 3.3372)	2.0099	0.983	(2.88924, 28.8176)	25.9283	0.726
(30, 13)	(0.06283, 1.4449)	1.3820	0.725	(1.38079, 3.2180)	1.8372	0.971	(2.93683, 31.6988)	28.762	0.713
(50, 13)	(0.05988, 1.4473)	1.3815	0.746	(1.41074, 3.1301)	1.7194	0.963	(2.97585, 31.9896)	29.0138	0.667

Table 6. Average estimates associated with a real data set under informative prior.

Parameter	Bayes								
	Loss function								
	SE	LINEX				GE			
		q = 0.5	q = −0.5	q = 1	q = −1	h = 0.5	h = −0.5	h = 1	h = −1
α_1	0.0945113	0.0822371	0.0914941	0.0853606	0.0950961	0.0939359	0.0948025	0.0942224	0.0945113
α_2	0.000259757	0.000235252	0.000253705	0.000241456	0.000259758	0.00025975	0.00025974	0.00025973	0.00025975
p	0.37502	0.347826	0.368551	0.354981	0.379717	0.370343	0.377351	0.372664	0.37502

Table 7. Average estimates associated with a real data set under a noninformative prior.

Parameter	Bayes								
	Loss function								
	SE	LINEX				GE			
		q = 0.5	q = −0.5	q = 1	q = −1	h = 0.5	h = −0.5	h = 1	h = −1
α_1	0.0863432	0.0740085	0.0833165	0.0771536	0.0868802	0.0858151	0.0866106	0.0860781	0.0863432
α_2	0.00024505	0.000220549	0.000239006	0.000226758	0.000245055	0.000245054	0.00024504	0.000245053	0.00024505
p	0.421053	0.388889	0.413511	0.397459	0.427176	0.41499	0.424107	0.418013	0.421053

Table 8. 95% Bayesian prediction bounds of and its lengths Ys, length of the Bayesian prediction in case informative prior.

(n, m)	Y1		Y4		Ym	
	(L, U)	Length	(L, U)	Length	(L, U)	Length
(29, 8)	(0.1684, 8.8445)	8.6761	(3.2331, 10.9303)	7.6572	(10.0454, 22.7782)	12.7328
	Y1		Y5		Ym	
(29, 10)	(0.1427, 8.2219)	8.0789	(3.7607, 10.8728)	7.1121	(10.3012, 23.791)	13.3263
	Y1		Y6		Ym	
(29, 12)	(0.1247, 7.5801)	7.4554	(4.1858, 10.8275)	6.6417	(10.477, 24.6225)	14.1455
	Y1		Y7		Ym	
(29, 13)	(0.1175, 7.2501)	7.1326	(5.0359, 10.9458)	5.9099	(10.527, 24.9788)	14.4518

Table 9. 95% Bayesian prediction bounds of and its lengths Ys, length of the Bayesian prediction in case noninformative prior.

(n, m)	Y1		Y4		Ym	
	(L, U)	Length	(L, U)	Length	(L, U)	Length
(29, 8)	(0.1652, 8.8044)	8.6392	(3.0659, 11.0204)	7.9545	(10.042, 25.5163)	15.4743
	Y1		Y5		Ym	
(29, 10)	(0.1401, 8.1364)	7.9963	(3.5348, 10.9614)	7.4266	(10.2933, 25.5688)	15.5755
	Y1		Y6		Ym	
(29, 12)	(0.1224, 7.4505)	7.3281	(3.9039, 10.9127)	7.0088	(10.5115, 27.5833)	17.0718
	Y1		Y7		Ym	
(29, 13)	(0.1153, 7.1035)	6.9882	(4.6317, 11.0395)	6.4078	(10.5868, 27.9938)	17.407

5.1. Numerical example

We provide a numerical example to demonstrate the methodology for the proposed estimates using real data. Assume classical real data in Keating [22] the set of times in operating days, between successive failures of air conditioning equipment aircraft, is given by $\{3.750, 0.417, 2.500, 7.750, 2.547, 2.042, 0.583, 1.000, 2.333, 0.833, 3.292, 3.500, 1.833, 2.458, 1.208, 4.917, 1.024, 6.500, 12.917, 3.167, 1.083, 1.833, 0.958, 2.583, 5.417, 8.667, 2.917, 4.208, 8.667\}$. We fitted the real data, and we noticed that it is fitted well with our proposed distribution, we also estimated the parameters, and the parameter came out with the following values ($p = 0.4232$, $\alpha_1 = 0.26907$, $\alpha_2 = 0.01, \theta = 1.2898$). The above type-I censored simulated this data sample of size ($n = 29$) from the mixture with the fixed $T = 3$, according type-I censored, it was found that $r = 17$. The type-I censored sample is given as follows: $r_1 = 0.417, 2.500, 2.547, 2.024, 0.583, 1.000, 2.333,$ and $r_2 = 0.833, 1.833, 2.458, 1.208, 1.024, 1.083, 1.833, 0.958, 2.583, 2.917$. Real data were used to calculate the Bayes estimates parameters α_1, α_2 and p (Tables 6 and 7) and the 95% Bayesian prediction interval (Tables 8 and 9).

5.2. Conclusion

This paper addresses the problems in estimating and predicting outcomes related to the mixture of Weibull and Gompertz distributions based on type-I censoring. We conclude by obtaining various estimators for the proposed parameters through Bayesian methods, considering both informative and noninformative prior distributions:

(a) Tables 1−3 show the performance of Bayes estimators obtained under informative prior has less MSE compared with the noninformative prior for all loss functions.
(b) For all estimators, it is observed that the value of the expected MSE of each estimator decreases as the sample size increases. Also, MSE of all estimators decreases with increase the value of T for fixed n. Moreover, for fixed n and T as increases, the MSE decreases as expected.
(c) Tables 4 and 5 illustrate that as the sample size increases, the lengths of Bayesian prediction intervals decrease. Consequently, when the confidence threshold is met, the Bayesian simulated coverage probability of Ys becomes one. Conversely, the length of Bayesian prediction intervals increases with s.
(d) In (Tables 8 and 9) we notice that the length of the Bayesian prediction intervals increases with increasing s, and also when we fix n with increasing m, the length of the intervals increases.

Ethics information

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Author contribution

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Conflicts of interest

There are no conflicts of interest.

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