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ORIGINAL ARTICLE – ASTRONOMY AND METEOROLOGY

Analysis of the Quasi-critical Inclination Under Third-body Perturbation

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Abstract

The critical inclination orbits are important and desirable for determining space missions. This inclination is a special value of the inclination of the orbital plane that makes the argument of the perigee staying constant, on average, when some perturbations are considered. We investigated the quasi-critical inclination problem in the current study. The considered perturbations include the oblateness and rotation of the planetary satellite (main body), as well as the gravitational influence of a third body. The third body is considered to move in a circular orbit. The equations of motion are formulated using the well-established Delaunay variables. Then the short and long-period terms are then eliminated using the Lie perturbation approach. Two canonical transformations are affected to obtain the normalized form of the Hamiltonian function of the dynamical system. Finally, we carried out several numerical explorations for the two planetary satellites Callisto and our Moon. The presence of the third celestial body significantly impacts the critical inclination value.

Keywords: Lie method, Planetary satellites, Quasi-critical inclination

1. Introduction

Researchers have spent decades examining in detail the motion of an artificial satellite about an oblate planet. Their primary attention was on the impact of the second zonal harmonic, or J_2, which results in a critical inclination value of $i = 63.43°$ [1–3]. Orlov [4] was the first person in history to identify the issue with critical inclination. That marked the start of this concept. Since certain space missions require this inclination, it is crucial concept to artificial satellite theory. Because the eccentricity and argument of perigee are often constant at critical inclination, less orbital maintenance procedures are needed for the orbits [5,6]. To prevent eccentricity variations and the rotation of the pericenter argument, critical inclination criteria are applied to artificial satellites orbiting the Earth, specifically the Tundra and Molniya orbits [7,8].

Numerous studies about the critical inclination problem were conducted with the assumption that the dominant term among the harmonic coefficients is the second zonal harmonic, denoted as J_2. The earlier presumption holds true for planets like Earth. In this instance, equatorial ellipticity perturbation, C_{22}, has no effect on the critical inclination. According to this conventional interpretation, critical inclination is found by equating the pericenter argument's variation to zero. Because eccentricity, semi-major axis, and inclination are the only factors that affect the fluctuation of the pericenter argument and are generally constant, this technique is entirely possible [3,5,9,10]. However, this criterion does not converge for some celestial bodies because of disturbances of the same order of magnitude as the oblateness term, J_2. Here, the inclination does not stay constant; instead, it performs long periodic oscillations. As a result, the idea of a critical inclination loses any relevance, and the idea of quasi-critical inclinations is presented [3].

Among the research published on this topic, Lara [11] studied it and found that, in addition to orbit

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size and shape, the degree of oblateness of the attracting celestial body affects the critical inclination value. Abd El-Salam and Abd El-Bar [12] obtained different families of critical inclinations for low-nearly circular lunar orbits. These orbits disappear rapidly when either the eccentricity axis or the semi-major changes slightly. Costa et al. [9] utilize an optimization approach to identify quasi-critical inclinations for both retrograde and direct orbits around the Jovian satellite Io.

In this work, we modify the work of the paper [9] and study the quasi-critical inclination problem under a third-body perturbation. The second zonal harmonic J_2 of the planetary satellite (main body), the main body's rotation, and the third-body attraction are all included in the force model's perturbations. We apply the Lie method to normalize the problem's Hamiltonian. Then we construct the canonical equations of the current dynamical model. We performed several numerical examples to show how the third body affects the critical inclination values.

2. Hamiltonian formalism

We consider the motion of a spacecraft around a planetary satellite (main body) under its zonal potential and the attraction induced by a third body. The spacecraft orbits the primary body in an elliptically inclined orbit with orbital elements $\{a, e, i, \omega, \Omega\}$. Where a is the semi-major axis, e the eccentricity, i the inclination, ω the argument of perigee, and Ω the longitude of the node of the orbiter, respectively. A synodic-rotating reference frame centered at the main body is introduced to eliminate the time-dependent terms in the Hamiltonian due to the third body. The x-axis continually points in the direction of the third body, and the basic reference plane aligns with the main body's equatorial plane. The third body is assumed to move along a circular equatorial orbit about the main body. Then, the Hamiltonian governing the dynamics of the spacecraft in terms of the canonical Delaunay variables is [13,14]

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_C + \mathcal{H}_Z + \mathcal{H}_{3b} \tag{1}$$

where $\mathcal{H}_K$ and $\mathcal{H}_C$ stands for the Kepler motion and the effect of Coriolis force respectively. Also, $\mathcal{H}_Z$ and $\mathcal{H}_{3b}$ for the gravitational potential of the planetary satellite and the third body attraction. The above terms in equation (1) are defined by

$$H_K = -\frac{\mu^2}{2L^2} \tag{2}$$

$$\mathcal{H}_C = -\omega_c H, \tag{3}$$

$$\mathcal{H}_Z = \frac{\mu}{r}\left(\frac{R_M}{r}\right)^2 J_2 P_2 \sin(\delta) \tag{4}$$

$$\mathcal{H}_{3b} = -\frac{k\,\mathcal{G}\,(m_0 + m_1)}{r_1} \sum_{n=2}^{\infty} \left(\frac{r}{r_1}\right)^n P_n(\cos\Psi) \tag{5}$$

where $k = \frac{m_1}{m_0 + m_1}$, and m_0, m_1 are the masses of the main and third bodies, respectively. R_M represents the radius of the planetary satellite and $\mu = Gm_0$ is its gravitational parameter. r_1 is the planetary satellite-third body distance and r is the radial distance of the orbiter concerning the main body, while ω_c is the rate of rotation of the central body. P_2 is the degree two of Legendre polynomial, δ is the spacecraft's latitude concerning the equatorial plane and Ψ is the angle created by the vectors of the third body and the spacecraft. To utilize perturbation theory, we express the current issue using simplistic action-angle variables $\{l, g, h, L, G, H\}$, where I, g, h are the mean anomaly, the argument of the perigee, the argument of the node in the rotating frame, respectively. The conjugate momenta variables are L, G, and H. These variables are functions of the orbital elements of the spacecraft and are defined in the rotating frame as follows

$$\begin{aligned} L &= \sqrt{\mu a}, \quad l = M \\ G &= L\sqrt{1-e^2}, \quad g = \omega \\ H &= G\cos i, \quad h = \Omega - \omega_c t \end{aligned} \tag{6}$$

It is noticeable that the order of magnitudes of the current perturbations is strongly dependent on the nature of the space mission. Consequently, for a spacecraft that moves around a planetary satellite, we may arrange the Hamiltonian of the problem according to the numerical values of mean motion values of the spacecraft, n, and the main body, ω_c. We assume that the ratio $\varepsilon = \frac{\omega_c}{n}$ is the small parameter of the problem and that $J_2 \sim \mathcal{O}\left(\frac{\omega_c}{n}\right)^2$. So, the Keplerian part $\mathcal{H}_\circ$ and Coriolis effect $\mathcal{H}_C$ are taken as the zero and the first-order terms, respectively. The perturbations $\mathcal{H}_{3b}$ and $\mathcal{H}_z$ are taken as the second order terms. Then we can write equation (1) as a power series in ε as

$$\mathcal{H} = \sum_{i \geq 0}^{2} \left(\frac{\varepsilon^i}{i!}\right) \mathcal{H}_i \tag{7}$$

Where $\mathcal{H}_\circ = \mathcal{H}_K, \mathcal{H}_1 = \frac{1}{\varepsilon}\mathcal{H}_C$ and $\mathcal{H}_2 = \frac{2}{\varepsilon^2}(\mathcal{H}_z + \mathcal{H}_{3b})$. Using spherical geometry, the angle Ψ is calculated by

$$\cos\Psi = \cos h\cos(\omega + f) - \cos i\sin h\sin(\omega + f)$$

Now we can write the Hamiltonian of the problem as

$$\mathcal{H}_K = -\frac{\mu^2}{2L^2} \quad (8)$$

$$\mathcal{H}_C = -\left(\frac{\mu}{a}\right)\left(\frac{\omega_c}{n}\right)\sqrt{1-e^2}\quad\cos i, \quad (9)$$

$$\mathcal{H}_Z = \left(\frac{\mu}{a}\right)\left(\frac{a}{r}\right)\left(\frac{R_M}{r}\right)^2 J_2 P_2(\sin i\sin(\omega + f)) \quad (10)$$

$$\mathcal{H}_{3b} = \frac{1}{2}\left(\frac{\mu}{a}\right)\left(\frac{\omega_c}{n}\right)^2\left(\frac{r}{a}\right)^2\Big[1 - 3(\cos h\cos(\omega + f) - \cos i\sin h\sin(\omega + f))^2\Big] \quad (11)$$

The periodic terms present in the above Hamiltonian function are averaged out using Lie-Hori method in the next section.

3. Perturbation approach

Canonical transformations have been extensively used to find solutions for Hamiltonian dynamical systems. In perturbation theory, one considers a Hamiltonian which consists of a solvable term and a small additional term. By small we mean that the effects of the term are small, so that, the motion with the perturbation does not differ greatly from the motion without the perturbation. If this is the case, one can hope to develop an integrable model for motion, a canonical transformation is employed. One of the most effective and methodical canonical methods for resolving canonical differential equation systems is the Lie transform. It can also be used for obtaining a deeper qualitative insight into the dynamical problem by studying the equilibria, phase flow evolution, and bifurcations [15].

A Lie series transformation φ of dimension m is defined as the solution.

$$x = x(y, Y, \varepsilon) = \sum_{i\geq 0}\frac{\varepsilon^i}{i!}x_i(y, Y),\ x_0 = (y, Y) = y \quad (12)$$

$$X = X(y, Y, \varepsilon) = \sum_{i\geq 0}\frac{\varepsilon^i}{i!}X_i(y, Y),\ X_0 = (y, Y) = Y \quad (13)$$

of the differential equations

$$\frac{dx}{d\varepsilon} = \nabla_X\mathcal{W}(y, Y, \varepsilon) \quad (14)$$

$$\frac{dX}{d\varepsilon} = -\nabla_x\mathcal{W}(y, Y, \varepsilon) \quad (15)$$

with the initial conditions

$$x(y, Y, \varepsilon = 0) = y, X(y, Y, \varepsilon = 0) = Y, where\ x, X, y, Y \in R^m$$

$$\mathcal{W} = \sum_{i\geq 0}\mathcal{W}_{i+1}(x, X, \varepsilon) \quad (16)$$

Where the transformation generator is denoted by $\mathcal{W}$. One canonical transformation that transforms a Hamiltonian is a Lie series transformation.

$$\mathcal{H}(x, X, \varepsilon) = \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{H}_{n,0}(x, X) \quad (17)$$

$$\mathcal{K}(y, Y, \varepsilon) = \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{H}_{n,0}(y, Y) \quad (18)$$

utilizing the relation

$$\mathcal{H}_{p,q} = \mathcal{H}_{p+1,q+1} + \sum_{k=0}^{p}\binom{p}{q}\{\mathcal{H}_{p-k,q-1}; \mathcal{W}_{k+1}\} \quad (19)$$

The Lie-Deprit technique looks for the generator of a Lie series transformation to change an old Hamiltonian (17) into a fully integrable one (18). Once the generating function $\mathcal{W}$, is obtained the classical equations can be used to directly determine the direct and inverse transformations of φ.

$$x = \grave{x} + \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{L}^n_{\mathcal{W}}(\grave{x}) \quad (20)$$

$$\grave{x} = x + \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{L}^n_{-\mathcal{W}}(x) \quad (21)$$

$$X = \grave{X} + \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{L}^n_{\mathcal{W}}(\grave{X}) \quad (22)$$

$$\grave{X} = X + \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{L}^n_{-\mathcal{W}}(X) \quad (23)$$

where $\mathcal{L}_{\mathcal{W}}(-)$ is the Poisson bracket $\{-, \mathcal{H}_0\}$.

4. Double-averaged problem

We can build the current analytical theory by using the perturbation technique based on the Hori-Lie transformation, with the Hamiltonian function stated as a power series of the small parameter ε.

Therefore, using the canonical transformations $\mathcal{M} : (l, g, h, L, G, H) \to (l', g', h', L', G', H')$ we find a new Hamiltonian function

$$\mathcal{H}' = \sum_{n=0}^{k-1} \frac{\varepsilon^n}{i!} \mathcal{H}_{\backslash}'(-, g', -, L', G', H') + \varepsilon^k \sum_{n \geq k} \frac{\varepsilon^{n-k}}{n!} \mathcal{H}_{\backslash}(l', g', h', L', G', H')$$

The usual procedure to solve our dynamical system is to transform the Hamiltonian of the problem to a new one in which all the angle variables are ignorable. We first use the classical Delaunay normalization to reduce the problem to two degrees of freedom. This process averages the problem over the mean anomaly l for each order of perturbation, and the fast-rotating terms are eliminated from the Hamiltonian function. In the transformation, we computed the generating function, $\mathcal{W}(l', g', h', L', G', H')$ of the transformation. It is to be noted that all the new variables are single-primed, but for the sake of simplicity of writing, we drop the primes. Then, up to the second order, we obtain

$$\mathcal{K}_0 = -\frac{\mu^2}{2L^2} \tag{24}$$

$$\mathcal{K}_1 = -\omega_c H \tag{25}$$

A further reduction is carried out to eliminate the argument of the perigee, ω. Then, the reduced Hamiltonian function depends only on the argument of the node and we obtained an integrable one-degree-of-freedom problem. Explicit expressions up to the second order of the new Hamiltonian are given by

$$\begin{aligned}\mathcal{KK} = &-\omega\rfloor\, H - \frac{\mu^2}{2L^2} - \frac{(G^2-3H^2)(G^3kL^5\omega c^2(3G^2-5L^2) - 4J2\mu^6R^2)}{16G^5\mu^2L^3} \\ &+ \left(\frac{3kL^2\omega c^2(G^2-H^2)(3G^2-5L^2)}{16G^2\mu^2} - \frac{45eHkL^7\omega c^3\sqrt{\frac{G^2(L^2-G^2)}{L^4}}}{16G\mu^4}\right)\cos(2h) \\ &- \frac{9\pi kL^5\omega c^3(G^2-H^2)(3G^2-5L^2)\sin(2h)}{16G^2\mu^4}\end{aligned} \tag{28}$$

$$\begin{aligned}\mathcal{K}_2 = &\frac{1}{32G^3\mu^4L^3}\Bigg(\frac{8J2\mu^8R^2(G^2-3H^2)}{G^2} + 2Gk\mu^2L^5\omega c^2\big(15(G-H)(G+H)(G-L)(G+L)\cos 2\omega \\ &+ 3\cos 2h \times 5(G^2+H^2)(G-L)(G+L)\cos 2\omega + (G-H)(G+H)(3G^2-5L^2) \\ &- (3G^2-5L^2)(G^2 - 6\,G\,H\sin 2h \sin 2\omega - 3H^2)\big) + 9GkL^8\omega c^3\Bigg(2GH\cos 2h\Bigg(2\pi(3G^2-5L^2)\sin 2\omega \\ &- 5\sqrt{1-\frac{G^2}{L^2}}\sqrt{G^2(L^2-G^2)}\Bigg) + \sin 2h\Bigg(-10\pi(G^2+H^2)(G-L)(G+L)\cos 2\omega - (G-H)(G+H) \\ &- 5\sqrt{1-\frac{G^2}{L^2}}\sqrt{G^2(L^2-G^2)}\sin 2\omega + 6\pi G^2 - 10\pi L^2\Bigg)\Bigg)\Bigg)\end{aligned} \tag{26}$$

$$\begin{aligned}\mathcal{W}_2 = &\frac{J\_2\mu^2R^2(f-M)(G^2-3H^2)}{8G^5} - \frac{J2\mu^2R^2\sqrt{\frac{L^2-G^2}{L^2}}(3G^2\cos 2\omega - 2G^2 - 3H^2\cos 2\omega + 6H^2)\sin f}{16G^5} \\ &- \frac{3J2\mu^2R^2(G^2-H^2)\cos 2\omega \sin 2f}{16G^5} - \frac{J2\mu^2R^2(G^2-H^2)\sqrt{\frac{L^2-G^2}{L^2}}}{16G^5} \times \cos 2\omega \sin 3f \\ &- \frac{3J2\mu^2R^2\cos f(G^2-H^2)\sqrt{\frac{L^2-G^2}{L^2}}\sin 2\omega}{16G^5} - \frac{3J2\mu^2R^2(G^2-H^2)\sin 2\omega\cos 2f}{16G^5} \\ &- \frac{J2\,\mu^2R^2(G^2-H^2)\sqrt{\frac{L^2-G^2}{L^2}}}{16G^5}\sin 2\omega \cos 3f\end{aligned} \tag{27}$$

Now the transformed Hamiltonian is of one degree of freedom in the longitude of the node, *h*, and there is no need for further normalization

5. Canonical equations

After ignoring the terms dependent on l and ω, the Hamiltonian of the problem is only a function of h, and the three action variables $L, G,$ and H (or $a, e,$ and i). Using Hamilton's equations, the mean rates of the Delaunay elements $(L_i,\ l_i)$ resulting from the third body attraction and the zonal potential are expressed as follows.

$$\dot{l}=\frac{\partial\mathcal{K}\mathcal{K}}{\partial L},\dot{L}=\frac{\partial\mathcal{K}\mathcal{K}}{\partial l} \tag{29}$$

To calculate the secular rates for the Delaunay elements, the following partial derivatives must be taken into account:

$$\frac{\partial}{\partial L}=\frac{\partial a}{\partial L}\frac{\partial}{\partial a}+\frac{\partial e}{\partial L}\frac{\partial}{\partial e} \tag{30}$$

$$\frac{\partial}{\partial G}=\frac{\partial a}{\partial G}\frac{\partial}{\partial a}+\frac{\partial e}{\partial G}\frac{\partial}{\partial e} \tag{31}$$

$$\frac{\partial}{\partial H}=\frac{\partial i}{\partial H}\frac{\partial}{\partial i} \tag{32}$$

The averaged canonical equations of motion of a spacecraft orbiting the primary body are obtained by using the above partial derivatives.

$$\begin{aligned}\dot{l}=&\frac{\mu^2}{L^3}-\frac{k\,L\,\omega c^2\left(G^2-3H^2\right)\left(3G^2-10L^2\right)}{8G^2\mu^2}-\frac{3\,\mathrm{J2}\,\mu^4R^2\left(G^2-3H^2\right)}{4G^5L^4}+\left(\frac{3kL\omega c^2(G-H)(G+H)\left(3G^2-5L^2\right)}{8G^2\mu^2}\right.\\&\left.-\frac{15\,kL^3\omega c^2(G-H)(G+H)}{8G^2\mu^2}-\frac{315\,e\,HkL^6\omega c^3\sqrt{\frac{G^2\left(L^2-G^2\right)}{L^4}}}{16G\mu^4}-\frac{45\,eHkL^7\omega c^3\left(\frac{2G^2}{L^3}-\frac{4G^2\left(L^2-G^2\right)}{L^5}\right)}{32G\mu^4\sqrt{\frac{G^2\left(L^2-G^2\right)}{L^4}}}\right)\cos 2h\\&-\frac{\sin 2h\left(45\pi k\omega c^3(G-H)(G+H)\left(3G^2L^4-7L^6\right)\right)}{16G^2\mu^4}\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{g}=&-\frac{6\mathrm{J2}\mu^6R^2\left(G^2-5H^2\right)+3G^3kL^5\omega c^2\left(G^4-5H^2L^2\right)}{8\,G^6\mu^2L^3}+\frac{3\,kL^2\omega c^2\left(\frac{15\,e\,G^2HL\,\omega c}{\sqrt{\frac{G^2\left(L^2-G^2\right)}{L^4}}}-\frac{10H^2\mu^2L^2}{G^3}+6\,G\mu^2\right)}{16\,\mu^4}\cos 2h\\&+\frac{9\pi kL^5\omega c^3\left(5H^2L^2-3G^4\right)}{8G^3\mu^4}\sin 2h\end{aligned} \tag{34}$$

$$\begin{aligned}\dot{h}=&\frac{G\,k\,L^5\omega c^2\left(3G^2-5L^2\right)(6H-6H\cos 2h)-\frac{24H\,\mathrm{J2}\,\mu^6R^2}{G^2}}{16G^3\mu^2L^3}-\omega c\\&+\frac{9kL^5\omega c^3\left(2\pi H\left(3G^2-5L^2\right)\sin 2h-5eGL^2\sqrt{\frac{G^2\left(L^2-G^2\right)}{L^4}}\cos 2h\right)}{16G^2\mu^4}\end{aligned} \tag{35}$$

$$\dot{L} = \frac{\partial \mathcal{K}\mathcal{K}}{\partial l} = 0 \tag{36}$$

$$\dot{G} = \frac{\partial \mathcal{K}\mathcal{K}}{\partial g} = 0 \tag{37}$$

$$\dot{H} = -\frac{9kL^5\omega c^3\left(10eGHL^2\sqrt{\frac{G^2\left(L^2-G^2\right)}{L^4}}\sin(2h) - 2\pi\left(G^2-H^2\right)\right)}{16G^2\mu^4}\left(3G^2-5L^2\right)\cos 2h + \frac{3kL^2\omega c^2\left(G^2-H^2\right)\left(3G^2-5L^2\right)\sin 2h}{8G^2\mu^2} \tag{38}$$

6. Quasi-critical inclinations

The constant value of inclination i from equation (34) for which the time derivative of the perigee g argument is zero is known as the critical inclination in the second zonal model. Which means that if an orbit has a critical inclination, then $\forall t \quad g(t) = g_0 \Leftrightarrow \dot{g} = 0$. This value is a fixed point for a phase portrait of (h, H). Nonetheless, equation provides the real variation of the perigee argument for the dynamical model under consideration (34). Thus, two curves may exist: one is constant, which is the desired curve and represents the condition for the existence of a critical inclination, $\dot{g} = 0$, and the one that describes the actual variation of the argument of perigee, equation (34). Here, zero is the only desired outcome for the time average of the difference between the intended and actual curves. for $T > 0$, and is finite [9]. Therefore,

$$\frac{1}{T}\int_{t_0}^{t_0+T}\left[\frac{dg}{dt}_{[actual]} - \frac{dg}{dt}_{[desired]}\right]dt = 0 \Leftrightarrow \int_{t_0}^{t_0+T}\left[\frac{dg}{dt}_{[actual]}\right]dt = 0 \tag{39}$$

In the actual variation case, the time evolution of H and h will change the inclination and the argument of perigee, so we cannot use the term 'critical 'value of inclination. Equation (39) has many problems, first, T is unknown. Second, equation (34) is coupled with equations (35) and (38). Finally, it is necessary to determine the values of the quantities h and H at each instant of time, to solve equation (39). To overcome such problems, we should first solve the equation coupling in (34). The Hamiltonian function, equation (28), defines implicitly $H(t)$ as a function of $h(t)$ by the function

$$H(t) = \xi(h(t)) = \frac{-c + \sqrt{c^2 + (c_1 - a_1\cos(2h))(b_1\cos(2h) + d)}}{b_1\cos(2h) + d} \tag{40}$$

where

$$a_1 = \frac{3\left(G^3kL^5\omega c^2\left(3G^2 - 5L^2\right) + 8\mu^6R^2\right)}{4G^3\mu^2L^3\varepsilon^2}$$

$$b_1 = \frac{3\left(G^3kL^5\omega c^2\left(3G^2 - 5L^2\right) + 8\mu^6R^2\right)}{4G^5\mu^2L^3\varepsilon^2}$$

$$c = \frac{\omega c}{\varepsilon}$$

$$d = \frac{-6G^3kL^5\omega c^2\left(3G^2 - 5L^2\right) - 24J_2\mu^6R^2}{8G^5\mu^2L^3\varepsilon^2}$$

$$c_1 = a_1\sin(2h) + b_1H^2\cos(2h) + 2cH + dH^2$$

Now if we replace equation (40) into equations (33)–(38), the system of ordinary differential equations is just a function of $h(t)$ and automatically decoupled. The system is separable since L and G are constants and g is a neglected Coordinate. Thus, equations (35) and (38) completely define h and H, and these variables in turn are used to obtain the time variation of l and g. Then, equations (35) and (38) constitute an independent system of differential equations from the rest of the equations:

$$\dot{h} = F_2(h, H), \dot{H} = F_3(h, H), [h(0), H(0)] = (h_0, H_0) \tag{41}$$

7. Numerical results

After the Lie method perturbation technique is used to eliminate short and long-period terms from the Hamiltonian of the problem, we are ready to study the quasi-critical inclination for a spacecraft orbiting planetary satellites. In our computations, we considered the gravity field of the main body up to the second zonal harmonic J_2, perturbations due to a third body that moves in a circular orbit, besides the rotation of the planetary satellite.

To explore the problem under concern let us consider the moderate-altitude semi-major axes values, for a satellite orbiting the planetary satellite Callisto, $a = 4000\ km$, $a = 4500\ km$, and $a = 5000\ km$. Figs. 1 and 2 show the inclination variations in the direct and retrograde cases, respectively. The simulations are performed for some selected semi-major axes around Callisto, with initial eccentricity $e = 0.03$ for direct orbit, and $e = 0.1$ for the retrograde one. The ascending node values are between $[0, 2\pi]$. Through several carried-out simulations, it can be noticed that direct and retrograde quasi-critical inclinations oscillate near the classical critical inclination value I_c (black line). The amplitude of variations does not exceed a few degrees around the classical value. It is observed that the quasi-critical inclinations have larger amplitudes in higher orbits above Callisto. This is because the orbits closest to

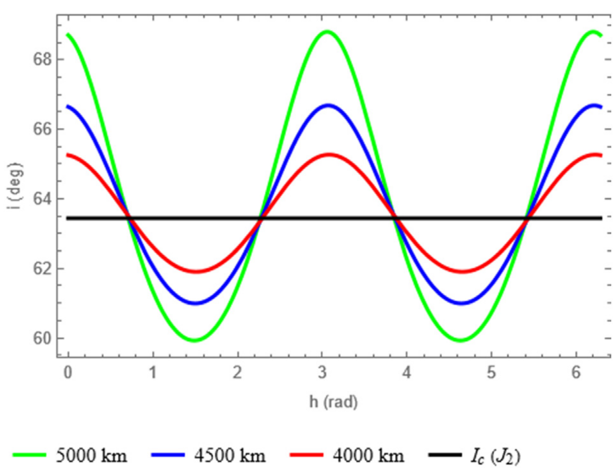

Fig. 1. Critical inclinations for Callisto direct orbits.

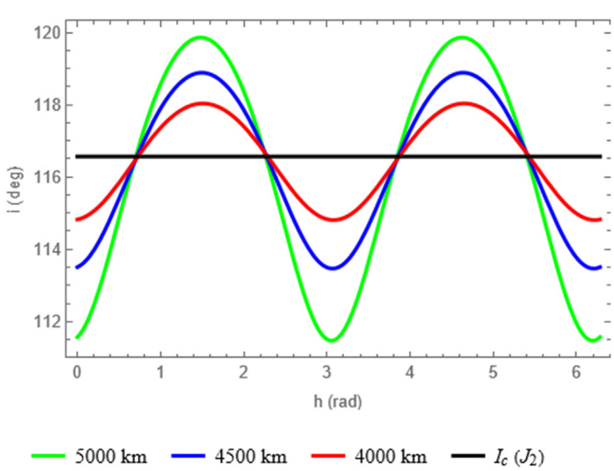

Fig. 2. Critical inclinations for Callisto retrograde orbits.

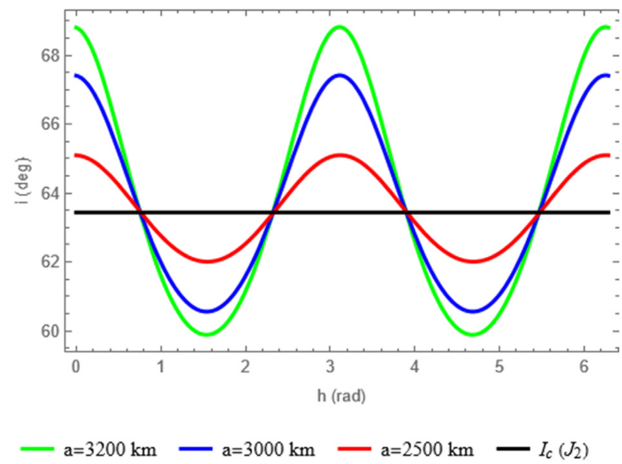

Fig. 3. Critical inclinations for Moon direct orbits.

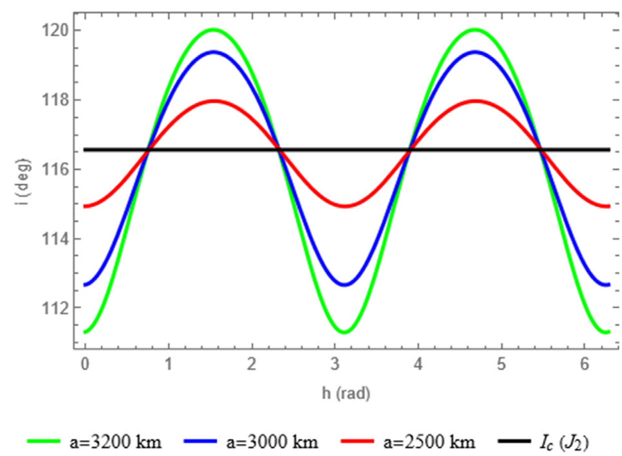

Fig. 4. Critical inclinations for Retrograde Moon orbits.

the third body are heavily impacted by its perturbation.

Similarly in Figs. 3 and 4, for a lunar orbiter, we considered some selected semi-major axes $a = 2500\ km$, $a = 3000\ km$, and $a = 3200\ km$. The initial eccentricity for both the direct and the retrograde orbits is $e = 0.03$. We also found that, as in Callisto's case, the variation in the inclination increases as the altitude of the lunar orbiter increases. As the altitude of the lunar orbiter increases more and more, the situation will be complicated, and the curves representing the variations will be discontinuous.

8. Conclusion

The issue of critical inclination is one of the important topics in the history of spacecraft dynamics. As a result, this topic has attracted the interest of numerous researchers who have investigated it from different perspectives. In the current work, we studied the critical inclination problem in the canonical perturbation theory frame of work.

We constructed the Hamiltonian function of the motion of a probe orbiting a planetary satellite in terms of the Delaunay variables. In addition to the third body's attraction, the oblateness of the planetary satellite is included in the force model. To simplify the dynamics of the problem, we doubly averaged the Hamiltonian by removing two of the angle variables. For example, we applied the current theory to a spacecraft moving around a planetary satellite. Several simulations are performed at different semi-major axes. We found that the critical inclination value is changed due to the perturbations considered. Moreover, we found that the variation in the critical inclination increases as the altitude of the orbiter increases.

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Author contributions

M. R., A. H. I., and G. F. M made conceptualization, visualization, and formal analysis; M.R., A. H. I. and A. G. A. A. achieved investigation and methodology; M.R., A. H. I., and A. G. A. A. made software; M. R., A. H. I., and G. F. M, wrote the main manuscript text original draft. All authors certify that they have participated sufficiently in the work to take public responsibility for the content.

Conflict of interest

There are no conflicts to declare.

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