Al-Azhar Bulletin of Science

Manuscript 1677

Section: Mathematics and Statistics

Some New Inequalities for C-Monotone Functions with Respect to ðq;uÞ-Hahn Difference Operator

Yasmen A. A. Elsaid

Afaf A. S. Zaghrout

Ahmed A. El-Deeb

ORIGINAL ARTICLE - MATHEMATICS AND STATISTICS

Some New Inequalities for C-Monotone Functions with Respect to (q,ω)-Hahn Difference Operator

Yasmen A.A. Elsaid [a], Afaf A.S. Zaghrout [a], Ahmed A. El-Deeb [b,*]

[a] Department of Mathematics, Faculty of Science (Girls), Al-Azhar University, Egypt
[b] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

Abstract

Motivated by certain results known from the literature, in our work we give several new inequalities for C-monotone functions with respect to (q,ω)-Hahn difference operator. If $\omega=0$, we obtain new inequalities for C-monotone functions. On the other hand, if $\omega=0, q\to 1$, our results reduce to integral inequalities known from the literature.

Keywords: C-Monotone functions, Hahn difference operator, Integral inequalities

1. Introduction

In 1995, Heinig and Maligranda [1], established that if $\nu\geq 0$ decreases on (a,b) and $\zeta\geq 0$ increases on (a,b) with $\zeta(a)=0$ and $-\infty\leq a<b\leq\infty$, then the inequality

$$\int_a^b \nu(\varsigma)d\zeta(\varsigma)\leq\left(\int_a^b \nu^\gamma(\varsigma)d[\zeta^\gamma(\varsigma)]\right)^{\frac{1}{\gamma}} \tag{1.1}$$

holds for every $\gamma\in(0,1]$, while for $1\leq\gamma<\infty$ the inequality is reversed. It has been also shown that when ν increases or (a,b) and ζ decreases on (a,b) with $\zeta(b)=0$, then holds the inequality

$$\int_a^b \nu(\varsigma)d[-\zeta(\varsigma)]\leq\left(\int_a^b \nu^\gamma(\varsigma)d[-\zeta^\gamma(\varsigma)]\right)^{\frac{1}{\gamma}} \tag{1.2}$$

where $\gamma\in(0,1]$. In the same paper, Heinig and Maligranda generalized (1.1) and established that when $0<p\leq q<\infty$, and f,g are non negative functions, then there is a constant $D>0$ since the inequality

$$\left[\int_0^\infty f(\varsigma)\nu^q(\varsigma)d\varsigma\right]^{\frac{1}{q}}\leq D\left[\int_0^\infty g(\varsigma)\nu^p(\varsigma)d\varsigma\right]^{\frac{1}{p}}, \tag{1.3}$$

relates all non negative decreased functions ν if and only if the inequality

$$\left[\int_0^\tau f(\varsigma)d\varsigma\right]^{\frac{1}{q}}\leq D\left[\int_0^\tau g(\varsigma)d\varsigma\right]^{\frac{1}{p}},$$

holds for all $\tau>0$. In addition, they also established that inequality (1.3) holds for all non negative increasing functions ν and $0<p\leq q<\infty$, if and only if the relation holds for all $\tau>0$.

$$\left[\int_\tau^\infty f(\varsigma)d\varsigma\right]^{\frac{1}{q}}\leq D\left[\int_\tau^\infty g(\varsigma)d\varsigma\right]^{\frac{1}{p}}.$$

In 1997, Pečarić *et al.* [2], extended the results of [1] to the case of C-monotone functions. Recall that if $s\leq\tau$ implies $\nu(\tau)\leq CV(s)$, for all $s,\tau\in[a,b]$, then ν is C-decreasing function on $[a,b]$. On the other hand, if $s\leq\tau$ implies $\nu(s)\leq CV(\tau)$ $s,\tau\in[a,b]$, then ν is C-increasing. Clearly, if $C=1$, then the notion of C-monotonicity reduces to the usual monotonicity. Some other type inequalities can be found in [3–6].

Pecarić *et al.* [2], generalized (1.1) and established that if $\theta:[0,\infty)\to\mathbb{R}$ is a concave, non negative and differentiable function such that $\theta(0)=0$, ν be C-decreasing with $C\geq 1$ and ζ increases on $[a,b]$, such that $\zeta(a)=0$, then holds the inequality

$$\theta\left(C\int_a^b \nu(\varsigma)d\zeta(\varsigma)\right)\leq C\int_a^b \theta'[\nu(\varsigma)\zeta(\varsigma)]\nu(\varsigma)d\zeta(\varsigma) \tag{1.4}$$

Received 8 May 2024; revised 23 May 2024; accepted 31 May 2024.
Available online 5 August 2024

* Corresponding author.
E-mail address: ahmedeldeeb@azhar.edu.eg (A.A. El-Deeb).

https://doi.org/10.21608/2636-3305.1677

In addition, let ν be C-increases with $C \geq 1$ and ζ increases on $[a, b]$, $\zeta(a) = 0$. Then, the following inequality holds

$$\theta\left(\frac{1}{C}\int_a^b \nu(\varsigma)d\zeta(\varsigma)\right) \geq \frac{1}{C}\int_a^b \theta'[\nu(\varsigma)\zeta(\varsigma)]\nu(\varsigma)d\zeta(\varsigma) \tag{1.5}$$

Analogous results have been derived for the case of a decreasing function. More precisely, let ν be C-increases, $C \geq 1$, ζ decreases on $[a,b]$ and $\zeta(b) = 0$. Then holds the inequality

$$\theta\left(C\int_a^b \nu(\varsigma)d[-\zeta(\varsigma)]\right) \leq C\int_a^b \theta'[\nu(\varsigma)\zeta(\varsigma)]\nu(\varsigma)d[-\zeta(\varsigma)] \tag{1.6}$$

while for C-decreasing function ν, $C \geq 1$, ζ decreases on $[a,b]$ and $\zeta(b) = 0$, holds the inequality

$$\theta\left(\frac{1}{C}\int_a^b \nu(\varsigma)d[-\zeta(\varsigma)]\right) \geq \frac{1}{C}\int_a^b \theta'[\nu(\varsigma)\zeta(\varsigma)]\nu(\varsigma)d[-\zeta(\varsigma)]. \tag{1.7}$$

In recent decades, great importance has been placed for proving discrete equivalents to the related continuous results in a variety of subjects of analysis. One of the reasons for the increasing attention in the discrete case is because discrete operators might behave quite differently than their continuous equivalent. In our work, we shall discrete inequalities as special cases of the results with the Hahn calculus.

Our purpose in this work is to expand inequality. (1.1), (1.2), (1.4), (1.5), (1.6) and (1.7) via (q, ω)-Hahn difference operator. In particular, we will employ the corresponding chain rule formula as well as the basic properties of C-Monotone functions. Of course, the results that we will derive contain classical results and the usual monotonicity.

The work is organized as follows: in Section 2, we give fundamental definitions and notions related to the Hahn calculus, which will be important in proving our major conclusions. In Section 3, we extend inequalities (1.1), (1.2), (1.4), (1.5), (1.6) and (1.7) via (q, ω)-Hahn difference operator. It turns out that if ω = 0 we obtain essentially new relations, while for ω = 0, $q \to 1$, our results reduce to inequalities presented in this introduction.

2. Preliminaries and lemmas

This section introduces the Hahn calculus, introduced in [7,8]. If $q \in (0,1)$, $\omega > 0$ be fixed, $\omega_0 := \omega/(1-q)$, and I be an interval of $\mathbb{R}$ contains ω_0. Consider $h(\tau) := q\tau + \omega, \tau \in I$. Both $h(\tau)$ and $h^{-1}(\tau)$. In the q-setting, the values ω_0 suggest that $\tau = 0$. In particular

$$\omega_0 = \lim_{k\to\infty}\underbrace{(h\circ h\circ\cdots\circ h)}_{k-\text{times}}(\tau).$$

The inverse $h(\tau)$ is $h^{-1}(\tau) = (\tau - \omega)/q, \tau \in I$. The q, ω-Hahn difference operator developed by Hahn in [9] may be presented as follows.

Definition 1. Consider u, a function defined on I. Hahn difference operator can be represented as follows:

$$D_{q,\omega}u(\tau) := \begin{cases} \dfrac{u(q\tau+\omega) - u(\tau)}{(qt+\omega) - t}, \tau \neq \omega_0, \\ u'(\omega_0), \tau = \omega_0. \end{cases}$$

Assume u is differentiable at ω_0. We refer to $D_{q,\omega}u$, as the q,ω-derivative of u, then u is q,ω-differentiable throughout I, if $D_{q,\omega}u(\omega_0)$ exists.

Note that

$$\lim_{\omega\uparrow 0} D_{q,\omega}u(\tau) = D_q u(\tau),$$

$$\lim_{q\downarrow 1,\omega\uparrow 0} D_{q,\omega}u(\tau) = u'(\tau).$$

Taking into consideration ↓ and ↑, which represents limitations from left and right at finite points. It is simple to demonstrate u, v is q,ω-differentiable at $\tau \in I$, then

$$D_{q,\omega}(\alpha u + \beta v)(\tau) = \alpha D_{q,\omega}u(\tau) + \beta D_{q,\omega}v(\tau), \alpha, \beta \in \mathbb{C},$$

$$D_{q,\omega}(uv)(\tau) = D_{q,\omega}(u(\tau))v(\tau) + u(q\tau+\omega)D_{q,\omega}g(\tau),$$

$$D_{q,\omega}\left(\frac{u}{v}\right)(\tau) = \frac{D_{q,\omega}(u(\tau))v(\tau) - u(\tau)D_{q,\omega}v(\tau)}{v(\tau)v(q\tau+\omega)}.$$

The last identity $v(\tau)v(q\tau+\omega) \neq 0$, can be obtained [7]. In [7], the right inverse for $D_{q,\omega}$ is defined in terms of Jackson-Nörlund sums as shown in [10]. Given a, $b \in I$, the q,ω -integral of u from a to b is defined as

$$\int_a^b u(\tau)d_{q,\omega}\tau := \int_{\omega_0}^b u(\tau)d_{q,\omega}\tau - \int_{\omega_0}^a u(\tau)d_{q,\omega}\tau,$$

$$\int_{\omega_0}^{\varsigma} u(\tau)d_{q,\omega}\tau := (\varsigma(1-q) - \omega)\sum_{k=0}^{\infty} q^k u\left(\varsigma q^k + \omega[k]_q\right), \varsigma \in I \tag{2.1}$$

Assuming that the sequence convergence at $\varsigma = a$ and $\varsigma = b$. The q-number is expressed as $[k]_q := \frac{1-q^k}{1-q}$, where $k \in \mathbb{N}_0$. In this situation, u is called q,ω-integrable on $[a,b]$. The sum on the right side of

(2.1) is called the Jackson-Nörlund sum. Refer to [10] to learn more about the relation between Nörlund sums and difference operators. According to [7], the basic theorem of q,ω-calculus states that let $u : I \to \mathbb{R}$ be continuous at ω_0,

$$U(\tau) := \int_{\omega_0}^{\tau} u(\varsigma) d_{q,\omega}\varsigma, \tau \in I$$

Then U be continuous at ω_0. Furthermore, $D_{q,\omega}U(\tau)$ exists for every $\tau \in I$ and

$$D_{q,\omega}U(\tau) = u(\tau)$$

Conversely

$$\int_a^b D_{q,\omega}u(\tau) d_{q,\omega}\tau = u(b) - u(a), \text{for all } a, b \in I.$$

Hence, the q, ω-integration by parts for continuous functions u, v are explained in [7,11].

$$\int_a^b u(\tau) D_{q,\omega}v(\tau) d_{q,\omega}\tau = u(\tau)v(\tau)\Big|_a^b - \int_a^b D_{q,\omega}(u(\tau)) v(q\tau + \omega) d_{q,\omega}\tau, a, b \in I$$

Lemma 1. [7]. If $s \in I, s > \omega_0, u$ *and* v *be* (q, ω)-integrable on I, then for a, $b \in \{sq^k + \omega[k]_q\}_{k=0}^{\infty}$, we have

$$\left| \int_a^b u(\tau) d_{q,\omega}\tau \right| \le \int_a^b |u(\tau)| d_{q,\omega}\tau$$

Consequently, if $|v(\tau)| \ge 0$ for all $\tau \in \{sq^k + \omega[k]_q\}_{k=0}^{\infty}$, then for all a, $b \in \{sq^k + \omega[k]_q\}_{k=0}^{\infty}$ the inequalities holds.

$$\int_{\omega_0}^b v(\tau) d_{q,\omega}\tau \ge 0 \quad \text{and} \quad \int_a^b v(\tau) d_{q,\omega}\tau \ge 0$$

Theorem 1. [12]. (Chain Rule involving Hahn-differential operator). *If* $v : I \to \mathbb{R}$ *be continuous and* q, ω-differentiable and $u : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then, there exists c between $q\tau + \omega$ and τ, we get

$$D_{q,\omega}(u \circ v)(\tau) = u'(v(c)) D_{q,\omega}v(\tau) \tag{2.2}$$

Lemma 2. *Let* u, $v : I \to \mathbb{R}$ be q, ω-integrable on $I, k \in \mathbb{R}$ and $a, b, c \in I$. Then

(i) $\int_a^a u(\tau) d_{q,\omega}\tau = 0$;
(ii) $\int_a^b ku(\tau) d_{q,\omega}\tau = k \int_a^b u(\tau) d_{q,\omega}\tau$;
(iii) $\int_a^b u(\tau) d_{q,\omega}\tau = -\int_b^a u(\tau) d_{q,\omega}\tau$;
(iv) $\int_a^b u(\tau) d_{q,\omega}\tau = \int_a^c u(\tau) d_{q,\omega}\tau + \int_c^b u(\tau) d_{q,\omega}\tau$;
(v) $\int_a^b (u(\tau) + v(\tau)) d_{q,\omega}\tau = \int_a^b u(\tau) d_{q,\omega}\tau + \int_a^b v(\tau) d_{q,\omega}\tau$.

The H ö lder's Inequality plays a fundamental role in the field of mathematics. Different variants can be found in [13−15]. The following inequality is known as H ö lder's Inequality involving Hahn calculus.

Theorem 2. [16]. *Let* $u, v : I \to \mathbb{R}$ *be* Hahn-integrable on I and $a, b \in I$. Then

$$\int_a^b \Big| u(\tau)v(\tau) \Big| d_{q,\omega}\tau \le \left\{ \int_a^b |u(\tau)|^j d_{q,\omega}\tau \right\}^{\frac{1}{j}} \times \left\{ \int_a^b |v(\tau)|^i d_{q,\omega}\tau \right\}^{\frac{1}{i}}, \tag{2.3}$$

where $j \ge 1, i = \dfrac{j}{j-1}$ *and* $\dfrac{1}{j} + \dfrac{1}{i} = 1$.

3. Main results

If no other information is explicitly mentioned, we suppose that all the functions are non-negative, continuous, q, ω-differentiable, and integral on I, on $[a, \infty)$.

In this section, $\theta : [0, \infty) \to \mathbb{R}$ refers to a non-negative, concave, and differentiable function with $\theta(0) = 0$. We are now ready to present and establish our main conclusions. Our initial conclusion extends inequalities (1.1) and (1.4) to consider the q, ω-Hahn difference operator.

Theorem 3. *Let I be any interval* $\tau \in I$ *and* $a, b \in [\tau q^m + \omega[m_q]]_{m=0}^{\infty}$, v be C-increasing on $[a, b]$ for $C \ge 1$, ζ increasing on interval $[a, b]$, and $\zeta(a) = 0$. Then

$$\theta\left(C \int_a^b \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] d_{q,\omega}\varsigma \right) \le C \int_a^b \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] \theta'[\nu(\varsigma)\zeta(\varsigma)] d_{q,\omega}\varsigma \tag{3.1}$$

Proof. Let

$$K = \theta\left(C \int_a^b \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] d_{q,\omega}\varsigma \right) - C \int_a^b \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] \theta'[\nu(\varsigma)\zeta(\varsigma)] d_{q,\omega}\varsigma \tag{3.2}$$

and

$$E(\tau) = C \int_a^{\tau} \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] d_{q,\omega}\varsigma \tag{3.3}$$

Therefore, we have from (3.2) and (3.3) that

$$K(\tau) = \theta(E(\tau)) - C \int_a^{\tau} \nu(\varsigma) \left[D_{q,\omega}\zeta(\varsigma) \right] \theta'[\nu(\varsigma)\zeta(\varsigma)] d_{q,\omega}\varsigma \tag{3.4}$$

Since v is C-increasing, then we have, for $\tau \geq \varsigma$, that $\nu(\tau) \leq C\nu(\varsigma)$, and then we obtain (note ζ is increasing and $\zeta(a) = 0$) that

$$\begin{aligned}\int_a^\tau C\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma &\geq \int_a^\tau \nu(\tau)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma \\ &= \nu(\tau)\int_a^\tau \left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma = \nu(\tau)[\zeta(\tau) - \zeta(a)] \\ &= \nu(\tau)\zeta(\tau).\end{aligned} \tag{3.5}$$

Substituting (3.3) into (3.5), we have

$$E(\tau) \geq \nu(\tau)\zeta(\tau). \tag{3.6}$$

Using the chain rule formulation (2.2) on the term $\theta(E(\tau))$, we see that there exists

$\eta \in [\tau, q\tau + \omega]$, such that

$$D_{q,\omega}\theta(E(\tau)) = \theta'(E(\eta))\left[D_{q,\omega}E(\tau)\right] \tag{3.7}$$

From (3.3), we obtain (note ζ is increasing) that

$$D_{q,\omega}E(\tau) = C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right] \geq 0 \tag{3.8}$$

and then E is increasing on $[a,b]$ and then we have, for $\eta \geq \tau$, that

$$E(\eta) \geq E(\tau) \tag{3.9}$$

When θ is concave on $[0,\infty)$, then $\theta'' < 0$ (θ' decreases on $[0,\infty)$) and thus, we notice from (3.9) that

$$\theta'(E(\eta)) \leq \theta'(E(\tau)) \tag{3.10}$$

Substitute (3.8) and (3.10) into (3.7) yields

$$D_{q,\omega}\theta(E(\tau)) \leq C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E(\tau)) \tag{3.11}$$

According to (3.6), $\theta'(E(\tau)) \leq \theta'(\nu(\tau)\zeta(\tau))$, and then we obtain (note ν is positive and ζ increases) that

$$C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E(\tau)) \leq C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)),$$

and thus we obtain from (3.11) that

$$D_{q,\omega}\theta(E(\tau)) \leq C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)) \tag{3.12}$$

Form (3.4), we have

$$D_{q,\omega}K(\tau) = D_{q,\omega}\theta(E(\tau)) - C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)) \tag{3.13}$$

Substitute (3.12), into (3.13), we have $D_{q,\omega}K(\tau) \leq 0$, and therefore K decreases on $[a,b]$. So $b > a$, we get $K(b) \leq K(a)$. Since $\theta(0) = 0$, we have from (3.2) that

$K(a) = \theta(0) = 0.$

Then $K(b) \leq 0$, and we obtain from (3.2), by putting $\tau = b$, then

$$\theta\left(\int_a^b C\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right) \leq C\int_a^b \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma,$$

that is the inequality (3.1).

Remark 1. *If* $\omega = 0$ *in* (3.1), *we obtain*

$$\theta\left(\int_a^b C\nu(\varsigma)\left[D_q\zeta(\varsigma)\right]d_q\varsigma\right) \leq C\int_a^b \nu(\varsigma)\left[D_q\zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_q\varsigma.$$

Remark 2. *If* $\omega = 0, q \to 1$ *in (3.1), we get the integral inequality (1.4) established by Pecaric′ et al.* [2]. Moreover, if $\omega = 0, q \to 1$. $C = 1, \theta(\tau) = \tau^p$ and $0 < p \leq 1$, our inequality (3.1) becomes integral inequality (1.1) established by Heinig and Maligranda [1].

Our next result is a dynamic extension of integral inequality (1.5) due to Pecarić *et al.* [2].

Theorem 4. *Let* I *be any interval* $\tau \in I$ *and* $a, b \in [\tau q^m + \omega[m_q]]_{m=0}^{\infty}$, ν be C-increasing on $[a,b]$, $C \geq 1$ and ζ increases on $[a,b]$, and $\zeta(a) = 0$. Then

$$\theta\left(\frac{1}{C}\int_a^b \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right) \geq \frac{1}{C}\int_a^b \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma. \tag{3.14}$$

Proof. Let

$$K(\tau) = \theta\left(\frac{1}{C}\int_a^\tau \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right) - \frac{1}{C}\int_a^\tau \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma, \tag{3.15}$$

and

$$E(\tau) = \int_a^\tau \frac{1}{C}\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma \tag{3.16}$$

Based on (3.15) and (3.16), we may conclude that

$$K(\tau) = \theta(E(\tau)) - \frac{1}{C}\int_a^\tau \nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right] \times \left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma. \tag{3.17}$$

Since ν C-increases, we have for $\varsigma \leq q\tau + \omega$, that $\nu(\varsigma) \leq C\nu(q\tau + \omega)$, then we have ($\zeta$ increases and $\zeta(a) = 0$), that

$$\int_a^{q\tau+\omega}\frac{1}{C}\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\geq\int_a^{q\tau+\omega}\nu(q\tau+\omega)\times\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma=\nu(q\tau+\omega)\int_a^{q\tau+\omega}\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma$$
$$=\nu(q\tau+\omega)\left[\zeta(q\tau+\omega)-\zeta(a)\right]$$
$$=\nu(q\tau+\omega)\zeta(q\tau+\omega),$$

and thus

$$\frac{1}{C}\int_a^{q\tau+\omega}\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\leq\nu(q\tau+\omega)\zeta(q\tau+\omega) \tag{3.18}$$

From (3.16), inequality (3.18) becomes

$$E(q\tau+\omega)\leq\nu(q\tau+\omega)\zeta(q\tau+\omega) \tag{3.19}$$

Using formula (2.2) for a term $\theta(E(\tau))$, we find that there exists $\eta\in[\tau,q\tau+\omega]$, such that

$$D_{q,\omega}\theta(E(\tau))=\theta'(E(\eta))\left[D_{q,\omega}E(\tau)\right] \tag{3.20}$$

From (3.16), we obtain (ζ increases) that

$$D_{q,\omega}E(\tau)=\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\geq 0 \tag{3.21}$$

and then E increases on $[a,b]$ and then we have $\eta\leq q\tau+\omega$, that

$$E(\eta)\leq E(q\tau+\omega) \tag{3.22}$$

Since θ is concave on $[0,\infty)$, then $\theta''<0$ (θ' decreases on $[0,\infty)$) and then, we observe from (3.22) that

$$\theta'(E(\eta))\geq\theta'(E(q\tau+\omega)) \tag{3.23}$$

Substituting (3.21) and (3.23) into (3.20), we get

$$D_{q,\omega}\theta(E(q\tau+\omega))\geq\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E(q\tau+\omega)) \tag{3.24}$$

From (3.19), we have that $\theta'(E(q\tau+\omega))\geq\theta'(\nu(q\tau+\omega)\zeta(q\tau+\omega))$, and then we have ($\nu$ be positive and ζ increases) that

$$\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E(q\tau+\omega))\geq\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(q\tau+\omega)\zeta(q\tau+\omega)),$$

thus we obtain from (3.24) that

$$D_{q,\omega}\theta(E(\tau))\geq\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(q\tau+\omega)+\zeta(q\tau+\omega)) \tag{3.25}$$

Form (3.17), we have

$$D_{q,\omega}K(\tau)=D_{q,\omega}\theta(E(\tau))-\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(q\tau+\omega)+\zeta(q\tau+\omega)) \tag{3.26}$$

Substituting (3.25) into (3.26), we find that $D_{q,\omega}K(\tau)\geq 0$, K increases on $[a,b]$. Since $b>a$, we may conclude that $K(b)\geq K(a)$. When $\theta(0)=0$, we have from (3.15) that

$$K(a)=\theta(0)=0.$$

Then $K(b)\geq 0$, and we obtain from (3.15), by putting $\tau=b$, that

$$\theta\left(\int_a^b\frac{1}{C}\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)\geq\frac{1}{C}\int_a^b\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma,$$

that is the desired inequality (3.14).

Remark 3. *If* $\omega=0$ *in* (3.14), *we obtain*

$$\theta\left(\int_a^b\frac{1}{C}\nu(\varsigma)\left[D_q\zeta(\varsigma)\right]d_q\varsigma\right)\geq\frac{1}{C}\int_a^b\nu(\varsigma)\left[D_q\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma)\zeta(q\varsigma)\right]d_q\varsigma.$$

On the other hand, if $\omega=0,q\to1$ *in* (3.14), *we obtain the integral inequality* (1.5) *proved by Pecarić et al.* [8].

In the sequel, we establish Hahn calculus versions of integral inequalities (1.2) and (1.6) from the Introduction.

Theorem 5. *Let* I *be any interval* $\tau\in I$ *and* a, $b\in[\tau q^m+\omega[m_q]]_{m=0}^{\infty}$, ν be C-increasing on $[a,b]$, $C\geq 1$, ζ decreases on $[a,b]$, and $\zeta(b)=0$. Then

$$\theta\left(C\int_a^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)\leq C\int_a^b\nu(\tau)\left[D_{q,\omega}-\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma. \tag{3.27}$$

Proof. Let

$$K(\tau)=-\theta\left(C\int_t^b\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)-C\int_t^b\nu(\varsigma)\left[D_{q,\omega}\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma, \tag{3.28}$$

and

$$E^*(\tau)=\int_t^b C\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]d_{q,\omega}\varsigma \tag{3.29}$$

From (3.28) and (3.29), we may conclude that

$$K(\tau)=-\theta(E^*(\tau))-C\int_t^b \nu(\varsigma)\left(D_{q,\omega}\zeta(\varsigma)\right)\theta'\left[\nu(q\tau+\omega)\right.\\ \left.\times\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma \tag{3.30}$$

For $q\tau+\ \omega\le\varsigma$, $v(q\tau+\omega)\le Cv(\varsigma)$ implies that ζ increases and $\zeta(b)\ =0$.

$$\begin{aligned}\int_{q\tau+\omega}^b C\nu(\varsigma)\left(D_{q,\omega}-\zeta(\varsigma)\right)d_{q,\omega}\varsigma &\ge \int_{q\tau+\omega}^b \nu(q\tau+\omega)\\ &\times\left(D_{q,\omega}-\zeta(\varsigma)\right)d_{q,\omega}\varsigma\\ &=\nu(q\tau+\omega)\int_{q\tau+\omega}^b\left(D_{q,\omega}-\zeta(\varsigma)\right)d_{q,\omega}\varsigma\\ &=\nu(q\tau+\omega)\left[-\zeta(b)+\zeta(q\tau+\omega)\right]\\ &=\nu(q\tau+\omega)\zeta(q\tau+\omega).\end{aligned} \tag{3.31}$$

Substituting (3.29) into (3.31), we get

$$E^*(q\tau+\omega)\ge\nu(q\tau+\omega)\zeta(q\tau+\omega) \tag{3.32}$$

Using formula (2.2) on the term $\theta(E^*(\tau))$, we find that there exists $\eta\in[\tau,q\tau+\omega]$, so that

$$D_{q,\omega}\theta(E^*(\tau))=\theta'(E^*(\eta))D_{q,\omega}E^*(\tau) \tag{3.33}$$

From (3.29), we get (ζ decreases) that

$$D_{q,\omega}E^*(\tau)=C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\le 0 \tag{3.34}$$

then $E^*(\tau)$ decreases on $[a,b]$ and then we have, for $\eta\le q\tau+\ \omega$, that

$$E^*(\eta)\ge E^*(q\tau+\omega) \tag{3.35}$$

Since θ is concave function on $[0,\infty)$, then $\theta''<0$ (θ' decreases on $[0,\infty)$) and then, we observe from (3.35) that

$$\theta'(E^*(\eta))\le\theta'\left(E^*(q\tau+\omega)\right) \tag{3.36}$$

Substituting (3.34) and (3.36) into (3.33), we get

$$D_{q,\omega}\theta\left(E^*(q\tau+\omega)\right)\ge C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'\left(E^*(q\tau+\omega)\right) \tag{3.37}$$

From (3.32), we have that $\theta'(E(q\tau+\omega))\le\theta'(\nu(q\tau+\omega)\zeta(q\tau+\omega))$, and then we get ($\nu$ be positive and ζ decreases) that

$$C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'\left(E^*(q\tau+\omega)\right)\\ \ge C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'\left(\nu(q\tau+\omega)\zeta(q\tau+\omega)\right),$$

thus we obtain from (3.37) that

$$D_{q,\omega}\theta\left(E^*(\tau)\ge C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'\left(v(q\tau+\omega)\zeta(q\tau+\omega)\right),\right.$$

and then

$$-D_{q,\omega}\theta(E^*(\tau))\le -C\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'\left(\nu(q\tau+\omega)\zeta(q\tau+\omega)\right) \tag{3.38}$$

From (3.30), we have

$$D_{q,\omega}K(\tau)=-D_{q,\omega}\theta(E^*(\tau))\\ +C\nu(\tau)D_{q,\omega}\zeta(\tau)\theta'\left(\nu(q\tau+\omega)\zeta(q\tau+\omega)\right) \tag{3.39}$$

Substituting (3.38) into (3.39), we get $D_{q,\omega}K(\tau)\le 0$, and K decreases on $[a,\ b]$, and $b>a$, we get $K(b)\le K(a)$. Since $\theta(0)\ =0$, we have from (3.28) that

$$K(b)=-\theta(0)=0,$$

then $K(a)\ge 0$, and we obtain from (3.28), by putting $\tau\ =a$, that

$$\theta\left(\int_a^b C\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)\\ \le C\int_a^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma+\omega)\zeta(q\varsigma+\omega)\right]d_{q,\omega}\varsigma.$$

That is the desired inequality (3.27).

Remark 4. If $\omega=0$ in (3.27), we obtain

$$\theta\left(\int_a^b C\nu(\varsigma)\left[D_q-\zeta(\varsigma)\right]d_q\varsigma\right)\\ \le C\int_a^b\nu(\varsigma)\left[D_q-\zeta(\varsigma)\right]\theta'\left[\nu(q\varsigma)\zeta(q\varsigma)\right]d_q\varsigma.$$

Remark 5. If $\omega=0, q\to 1$ in (3.27), we obtain the inequality (1.6) established by *Pecaric′* et al. [8]. *In addition, if $C\ =1$, and $\theta(\tau)=\tau^p$ and $0<p\le 1$, we obtain the integral inequality (1.2) established by Heinig and Maligranda* [1].
To finish our discussion, we can extension inequality (1.7).
Theorem 6. Let I be any interval $\tau\in I$ and a, $b\in[\tau q^m+\omega[m_q]]_{m=0}^{\infty}$, ν be C-decreasing on $[a,b]$, $C\ge 1$ and ζ decreases on $[a,b]$ and $\zeta(b)=0$. Then

$$\theta\left(\frac{1}{C}\int_a^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)\\ \ge\frac{1}{C}\int_a^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma. \tag{3.40}$$

Proof. Let

$$K(\tau)=-\theta\left(\frac{1}{C}\int_t^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]d_{q,\omega}\varsigma\right)\\ +\frac{1}{C}\int_t^b\nu(\varsigma)\left[D_{q,\omega}-\zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma, \tag{3.41}$$

and

$$E^*(\tau) = \int_t^b \frac{1}{C}\nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma \tag{3.42}$$

Based on (3.41) and (3.42), we may conclude that

$$K(t) = -\theta(E^*(\tau)) + \frac{1}{C}\int_t^b \nu(\varsigma)\big(D_{q,\omega} - \zeta(\varsigma)\big)\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma \tag{3.43}$$

Given ν be C-decreases, for $\varsigma \geq \tau$, we have $\nu(\varsigma) \leq C\nu(\tau)$, Consequently, considering that (decreases and $\zeta(b) = 0$), we obtain

$$\begin{aligned}\int_t^b \frac{1}{C}\nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma &\geq \int_t^b \nu(\tau)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma \\ &= \nu(\tau)\int_t^b \left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma = \nu(\tau)[-\zeta(b) + \zeta(\tau)] \\ &= \nu(\tau)\zeta(\tau),\end{aligned} \tag{3.44}$$

and then

$$\frac{1}{C}\int_t^b \nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma \leq \nu(\tau)\zeta(\tau) \tag{3.45}$$

Substituting (3.42) into (3.45), we get

$$E^*(\tau) \leq \nu(\tau)\zeta(\tau) \tag{3.46}$$

Using the chain rule formula (2.2) on the expression $\theta(E^*(\tau))$, we observe that there exists $d \in [\tau, q\tau + \omega]$, so that

$$D_{q,\omega}\theta(E^*(\tau)) = \theta'(E^*(d))\left[D_{q,\omega}E^*(\tau)\right] \tag{3.47}$$

From (3.42), we obtain (ζ decreases) that

$$D_{q,\omega}E^*(\tau) = \frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right] \leq 0, \tag{3.48}$$

Given that $E^*(\tau)$ decreases on $[a,b]$, for $d \geq \tau$, we have

$$E^*(d) \leq E^*(\tau) \tag{3.49}$$

Furthermore, since θ is concave on $[0,\infty)$, implying $\theta'' < 0$ (meaning θ' decreases on $[0, \infty)$), we can observe from (3.49) that

$$\theta'(E^*(d)) \geq \theta'(E^*(\tau)) \tag{3.50}$$

Substituting (3.48) and (3.50) into (3.47), we get

$$D_{q,\omega}\theta(E^*(\tau)) \leq \frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E^*(\tau)) \tag{3.51}$$

From (3.46), we have that $\theta'(E^*(\tau)) \geq \theta'(\nu(\tau)\zeta(\tau))$, and then we obtain (ν be positive and ζ decreases) that

$$\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(E^*(\tau)) \leq \frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)),$$

thus we obtain from (3.51) that

$$D_{q,\omega}\theta(E^*(\tau)) \leq \frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)),$$

and then

$$-D_{q,\omega}\theta(E^*(\tau)) \geq -\frac{1}{C}\nu(\tau)\left[D_{q,\omega}\zeta(\tau)\right]\theta'(\nu(\tau)\zeta(\tau)) \tag{3.52}$$

From (3.43), we have

$$D_{q,\omega}K(\tau) = -D_{q,\omega}\theta(E^*(\tau)) + \frac{1}{C}\nu(\tau)D_{q,\omega}\zeta(\tau)\theta'(\nu(\tau)\zeta(\tau)) \tag{3.53}$$

Substituting (3.52) into (3.53), we note that $D_{q,\omega}K(\tau) \geq 0$, and K increases on $[a,b]$. When $b > a$, we see that $K(b) \geq Ka)$. Since $\theta(0) = 0$, we have from (3.41) that

$$K(b) = -\theta(0) = 0,$$

then $K(a) \leq 0$, we get from (3.41), by putting $\tau = a$, that

$$-\theta\left(\frac{1}{C}\int_a^b \nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma\right) + \frac{1}{C}\int_a^b \nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma \leq 0,$$

and thus

$$\theta\left(\frac{1}{C}\int_a^b \nu(\tau)\left[D_{q,\omega} - \zeta(\varsigma)\right]d_{q,\omega}\varsigma\right) \geq \frac{1}{C}\int_a^b \nu(\varsigma)\left[D_{q,\omega} - \zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_{q,\omega}\varsigma,$$

that is the desired inequality (3.40).

Remark 6. If $\omega \to 0$ in (3.40), we obtain

$$\theta\left(\frac{1}{C}\int_a^b \nu(\varsigma)\left[D_q - \zeta(\varsigma)\right]d_q\varsigma\right) \geq \frac{1}{C}\int_a^b \nu(\varsigma)\left[D_q - \zeta(\varsigma)\right]\theta'[\nu(\varsigma)\zeta(\varsigma)]d_q\varsigma.$$

Furthermore, if $\omega \to 0, q \to 1$ in (3.40), we obtain inequality (1.7) established by *Pecaric′* et al. [2].

4. Conclusion

In this manuscript we discussed some new investigations of the Hardy inequality and its companion inequalities via (q, ω)-Hahn difference operator. These inequalities have certain conditions that have not been studied before. For example, in Theorem 3, we are dealing with several new inequalities for C-monotone functions concerning (q, ω)-Hahn difference operator. Besides that, to obtain some new inequalities as special cases, we also extended our inequalities to continuous calculus.

Funding

The authors did not receive any funding from their institutes for this work.

Author's contributions

Y.A.E and A.A.E.-D. wrote the main manuscript text and A.A.S.Z. prepare the formal analysis.
All authors reviewed the manuscript.

Conflict of interest

There are no conflicts of interest.

References

[1] Heinig H, Maligranda L. Weighted inequalities for monotone and concave functions. Stud Math 1995;116:133–65.
[2] Pecaric J, Peric I, Persson LE. Integral inequalities for monotone functions. J Math Anal Appl 1997;215:235–51.
[3] Ali MH, El-Owaidy HM, Ahmed HM, El-Deeb AA, Samir I. Optical solitons and complexitons for generalized Schrödinger–Hirota model by the modified extended direct algebraic method. Opt Quant Electron 2023;55:675.
[4] El-Deeb AA, Elsennary HA, Baleanu D. Some new Hardy-type inequalities on time scales. Adv Diff Equat 2020 2020: 1–21.
[5] El-Deeb AA, El-Sennary HA, Khan ZA. Some reverse inequalities of Hardy type on time scales. Adv Diff Equat 2020 2020:1–18.
[6] El-Deeb AA, El-Sennary HA, Khan ZA. Some Steffensen-type dynamic inequalities on time scales. Adv Diff Equat 2019 2019:1–14.
[7] Annaby MH, Hamza AE, Aldwoah KA. Hahn difference operator and associated jackson– norlund integrals. J Optim Theor Appl 2012;154:133–53.
[8] Hahn W. Uber orthogonalpolynome, die q-differenzengleichungen ¨ genügen. Math Nachr 1949;2:4–34.
[9] Hahn W. Ein beitrag zur theorie der orthogonalpolynome. Monatshefte fur mathematik 1983;95:19–24.
[10] Jordan C. Calculus of finite differences. New York: Comp: chelsea publ; 1965.
[11] Aldwoah KA. Generalized time scales and associated difference equations. PhD thesis, Ph. D. thesis. Cairo University; 2009.
[12] Bolster D, de Anna P, Benson DA, Tartakovsky AM. Incomplete mixing and reactions with fractional dispersion. Adv Water Resour 2012;37:86–93.
[13] Callebaut DK. Generalization of the cauchy-schwarz inequality. J Math Anal Appl 1965;12:491–4.
[14] Riesz F, Nagy BS. Functional analysis. Mineola, N. Y., United States of America.: Dover Publications, Inc., Courier Corporation; 2012.
[15] Wu S. A new sharpened and generalized version of holder's inequality and its applications. Appl Math Comput 2008;197: 708–14.
[16] Nosheen A, Saba A, Ali Khan K, Kahungu MK. h-opial-type inequalities via hahn operators. Discrete Dynam Nat Soc 2022;2022.