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Convolution Properties for a Class Defined by q-Al-Oboudi-Derivative Operator

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Abstract

In this paper, and using q-Al-Oboudi derivative operator and the concept of subordinating, we define a new class of univalent functions and investigate convolution properties, the necessary and sufficient condition and coefficient estimates for functions in this class. Our results generalizes previous results.

Keywords: Analytic functions, Coefficient estimates, Convolution properties, Hadamard product, q-difference operator, Univalent functions

1. Introduction

Let £ denote the class of analytic functions in U

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k, U=\{z\in C: |z|<1\} \tag{1}$$

For f, g analytic in U, f is subordinate to g, $(f \prec g)$ if $\exists$ an analytic function ω, with $\omega(0)=0$ and $|\omega(z)|<1 (z\in U)$, such that $f(z)=G(\omega(z))$. Furthermore, if g is univalent in U, then the equivalence:

$$f(z)\prec g(z) \Leftrightarrow f(0)=g(0), f(U)\subset g(U),$$

holds (see Refs. [4,7]).

Let $S(\alpha)\subset £$ $(0\le\alpha<1)$ consisting of all starlike functions of order α, in U.

The Hadamard product of E given by (1), and $g(z)=z+\sum_{k=2}^{\infty} b_k z^k$ is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty} a_k b_k z^k=(g*f)(z) \tag{2}$$

Let $S^*_{q,\zeta,\vartheta}$ $(-1\le\vartheta<\zeta\le 1)$ be the subclass of £ which defined by see [10]

$$S^*_{q,\zeta,\vartheta}=\left\{f\in £ : \frac{z\partial_q f(z)}{f(z)}\prec\frac{1+\zeta z}{1+\vartheta z}, z\in U\right\}$$

where

$$\partial_q f(z)=\begin{cases}\dfrac{f(z)-f(qz)}{(1-q)z} & for\ z\neq 0\\ f'(0) & for\ z=0,\end{cases}$$

$$=1+\sum_{k=2}^{\infty}[k]_q a_k z^{k-1} \tag{3}$$

which was defined by Refs. [5,6,8,11] see also Ref. [3] and

$$[k]_q=\frac{1-q^k}{1-q}$$

For $N=\{1,2,\ldots\}$, $\lambda\ge 0$ and f given by (1), Aouf *et al.* [1] defined the generalized q-Sălăgean operator by

$$D^0_{q,\lambda}f(z)=f(z)$$

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$$D^1_{q,\lambda}f(z) = (1-\lambda)f(z) + \lambda z(\partial_q f(z))$$

$$= z + \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]a_k z^k$$

$$\vdots$$

$$D^n_{q,\lambda}f(z) = (1-\lambda)D^{n-1}_{q,\lambda}f(z) + \lambda z\partial_q\left(D^{n-1}_{q,\lambda}f(z)\right)$$

$$= z + \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n a_k z^k \tag{4}$$

which for $q \to 1^-$ reduces to D^n_λ that in turn for $\lambda = 1$ tends to Sălăgean operator (see Refs. [2,9]).

Motivating the class defined by Seoudy and Aouf [10], we define the following class.

Making use of $D^n_{q,\lambda}f(z)$ we introduce the subclass $S^{n*}_{q,\zeta,\vartheta}(\lambda)$ of the class £. For $q \in (0,1), \lambda \geq 0, -1 \leq \vartheta < \zeta \leq 1$, let

$$S^{n*}_{q,\zeta,\vartheta}(\lambda) = \left\{f \in £ : D^n_{q,\lambda}f(z) \in S^*_{q,\zeta,\vartheta}(\lambda)\right\} \tag{5}$$

Note that

(1) $S^0_{q,\zeta,\vartheta}(\lambda) = S^*_{q,\zeta,\vartheta}$ (Seoudy and Aouf [10]);

(2) $S^n_{q,1-2\alpha,-1}(\lambda) = S^n_q(\alpha,\lambda)$ $(0 \leq \alpha < 1, \lambda \geq 0)$;

$$S^n_q(\alpha,\lambda) = \left\{f \in £ : Re\frac{z\partial_q D^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} > \alpha, z \in U\right\}$$

(3) $S^n_{q,(1-2\alpha)\beta,-\beta}(\lambda) = S^n_q(\lambda,\alpha,\beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1, \lambda \geq 0)$

$$S^n_q(\lambda,\alpha,\beta) = \left\{f \in £ : \left|\frac{\frac{zD^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} - 1}{\frac{zD^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} + 1 - 2\alpha}\right| < \beta, z \in U\right\}$$

(4) $\lim_{q\to 1} S^n_{q,\zeta,\vartheta}(\lambda) = \left\{f \in £ : \frac{z\left(D^n_\lambda f(z)\right)'}{D^n_\lambda f(z)} \prec \frac{1+\zeta z}{1+\vartheta z}, z \in U\right\} = S^n_{\zeta,\vartheta}(\lambda)$

2. Convolution properties

Otherwise mentioned, let $\theta \in [0,2\pi), 0 < q < 1, \lambda \geq 0, n \in N_0 = N \cup \{0\}$ and $-1 \leq \vartheta < \zeta \leq 1$.

Theorem 2.1. If $f \in S^{n*}_{q,\zeta,\vartheta}(\lambda) \Leftrightarrow$

$$\frac{1}{z}\left[D^n_{q,\lambda}f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0 \ (z \in U) \tag{6}$$

$$\forall\ L = L_\theta = \frac{e^{-i\theta} + \zeta}{\zeta - \vartheta} \text{ and also } L = 1.$$

Proof: First suppose $f \in S^{n*}_{q,\zeta,\vartheta}(\lambda)$, that is

$$\frac{z\partial_q D^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} \prec \frac{1+\zeta x}{1+\vartheta x} \tag{7}$$

Since $f(z) \neq 0, z \in U^* = U/\{0\}$; then $\left(\frac{1}{z}\right)f(z) \neq 0$, and this is equivalent to the fact that (6) holds for $L = 1$. From (7) there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$, $|\omega(z)| < 1$ such that

$$\frac{z\partial_q D^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} = \frac{1+\zeta w(z)}{1+\vartheta w(z)} \tag{8}$$

that is

$$\frac{z\partial_q D^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)} \neq \frac{1+\zeta e^{i\theta}}{1+\vartheta e^{i\theta}} \tag{9}$$

or

$$\frac{1}{z}\left[(1+\vartheta e^{i\theta})z\partial_q D^n_{q,\lambda}f(z) - (1+\zeta e^{i\theta})D^n_{q,\lambda}f(z)\right] \neq 0 \tag{10}$$

Since,

$$D^n_{q,\lambda}f(z) * \frac{z}{1-z} = D^n_{q,\lambda}f(z),$$

$$D^n_{q,\lambda}f(z) * \frac{z}{(1-z)(1-qz)} = z\partial_q D^n_{q,\lambda}f(z),$$

we may write (10) as

$$\frac{1}{z}\left[D^n_{q,\lambda}f(z) * \left(\frac{(1+\vartheta e^{i\theta})z}{(1-z)(1-qz)} - \frac{(1+\zeta e^{i\theta})z}{1-z}\right)\right]$$

$$= \frac{(\vartheta-\zeta)e^{i\theta}}{z}\left[D^n_{q,\lambda}f(z) * \frac{z - \frac{\left(e^{-i\theta}+\zeta\right)qz^2}{(\zeta-\vartheta)}}{(1-z)(1-qz)}\right] \neq 0 \tag{11}$$

which leads to (6).

Reversely, because assumption (6) holds for $L = 1$, it follows that $\left(\frac{1}{z}\right)f(z) \neq 0$, for all $z \in U$; hence, $\Phi(z) = \frac{z\partial_q D^n_{q,\lambda}f(z)}{D^n_{q,\lambda}f(z)}$ is analytic in U. From the proof of the first part, we see that (6) is equivalent to (9), that is

$$\frac{z\partial_q D^n_{q,\lambda} f(z)}{D^n_{q,\lambda} f(z)} \neq \frac{1+\zeta e^{i\theta}}{1+\vartheta e^{i\theta}} \tag{12}$$

Putting

$$\Psi(z) = \frac{1+\zeta z}{1+\vartheta z} \tag{13}$$

in (11) we observe that for all $z \in U$, Φ and Ψ not intersect. Thus the connected component $\Phi(U) \subset C \setminus \Psi(\partial U)$. Since $\Phi(0) = \Psi(0)$ and Ψ is univalent, then $\Phi(z) \prec \Psi(z)$, that is (7), holds, i. e $E \in S^n_{q,\zeta,\vartheta}(\lambda)$. Putting $n = 0$ in theorem 2.1, we have [Ref. [8], theorem 1].

Corollary 2.1. the function $f \in S^{0*}_{q,\zeta,\vartheta}(\lambda) \Leftrightarrow$

$$\frac{1}{z}\left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0 \ (z \in U) \tag{14}$$

$\forall\ L = L_\theta = \dfrac{e^{-i\theta}+\zeta}{\zeta - \vartheta}$ and also $L = 1$.

Theorem 2.2. The function $f \in S^{n*}_{q,\zeta,\vartheta}(\lambda) \Leftrightarrow$

$$1 - \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta}{\zeta - \vartheta} a_k z^{k-1} \neq 0 \tag{15}$$

Proof: From theorem 2.1, we find that $f \in S^{n*}_{q,\zeta,\vartheta}(\lambda)$ if and only if (6) holds, for all $L = L_\theta = \frac{e^{-i\theta}+\zeta}{\zeta-\vartheta}$ and also $L = 1$. The left-hand side of (6) can be written as

$$\frac{1}{z}\left[D^n_{q,\lambda} f(z) * \left(\frac{z}{(1-z)(1-qz)} - \frac{Lqz^2}{(1-z)(1-qz)}\right)\right]$$

$$= \frac{1}{z}\left\{z\partial_q D^n_{q,\lambda} f(z) - L\left[z\partial_q D^n_{q,\lambda} f(z) - D^n_{q,\lambda} f(z)\right]\right\} \tag{16}$$

$$= 1 - \sum_{k=2}^{\infty}\left([k]_q(L-1) - L\right)\left[1+\lambda\left([k]_q - 1\right)\right]^n a_k z^{k-1}$$

$$= 1 - \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta}{\zeta - \vartheta} a_k z^{k-1}$$

3. Coefficient estimates

As an application of theorem 2.2, we next determine coefficient estimate and inclusion property for a function to be in the class $S_{q,\zeta,\vartheta^{n*}(\lambda)}$

Theorem 3.1. If the function f satisfies:

$$\sum_{k=2}^{\infty}\left([k]_q(1-\vartheta) - 1 + \zeta\right)\left[1+\lambda\left([k]_q - 1\right)\right]^n |a_k| \leq \zeta - \vartheta,$$

then $f \in S^{n*}_{q,\zeta,\vartheta}(\lambda)$

Proof: Since

$$\left|1 - \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta}{\zeta - \vartheta} a_k z^{k-1}\right|$$

$$> 1 - \sum_{k=2}^{\infty}\left|\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta}{\zeta - \vartheta}\right| |a_k|$$

$$= 1 - \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{\left|[k]_q(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta\right|}{\zeta - \vartheta} |a_k|$$

$$> 1 - \sum_{k=2}^{\infty}\left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(1-\vartheta) - 1 + \zeta}{\zeta - \vartheta} |a_k| > 0,$$

the result follows from theorem 2.2.

Taking $q \to 1^-$ in theorems 2.1, 2.2, and 3.1, we obtain.

Corollary 3.1. the function $f \in S^{n*}_{\lambda}(\alpha)(0 \leq \alpha < 1) \Leftrightarrow$

$$\frac{1}{z}\left[D^n_\lambda f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \neq 0 \ (z \in U)$$

$\forall\ L = L_\theta = \frac{e^{-i\theta}+\zeta}{\zeta-\vartheta}$ and also $L = 1$.

Corollary 3.2. the function $f \in S^{n*}_{\zeta,\vartheta}(\lambda) \Leftrightarrow$

$$1 - \sum_{k=2}^{\infty}[1+\lambda(k-1)]^n \frac{k(e^{-i\theta}+\vartheta) - e^{-i\theta} - \zeta}{\zeta - \vartheta} a_k z^{k-1} \neq 0.$$

Corollary 3.3. the function $f \in S^{n*}_{\zeta,\vartheta}(\lambda) \Leftrightarrow$

$$\sum_{k=2}^{\infty}(k(1-\vartheta) - 1 + \zeta)[1+\lambda(k-1)]^n |a_k| \leq \zeta - \vartheta.$$

Putting $\zeta = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $\vartheta = -1$ in Theorems 2.1 and 2.2, we obtain.

Corollary 3.4. the function $f \in S^{n*}_q(\alpha,\lambda)(0 \leq \alpha < 1) \Leftrightarrow$

$$\frac{1}{z}\left[D^n_{q,\lambda} f(z) * \frac{z - Mqz^2}{(1-z)(1-qz)}\right] \neq 0 \ (z \in U)$$

$$\forall\ M = M_\theta = \frac{(e^{-i\theta}+1-2\alpha)}{2(1-\alpha)}, 0 \leq \alpha < 1, \text{and also}\ M = 1.$$

Corollary 3.5. the function $z \in S_q^{n*}(\alpha,\lambda) \Leftrightarrow$

$$1 - \sum_{k=2}^{\infty} \left[1+\lambda\left([k]_q - 1\right)\right]^n \frac{[k]_q(e^{-i\theta}-1) - e^{-i\theta} - 1 + 2\alpha}{2(1-\alpha)} a_k z^{k-1} \neq 0$$

Remark. Putting n=0 in theorems 2.1, 2.2, and 3.1, we obtain the corresponding results obtained by Seoudy and Aouf [10].

4. Conclusion

Throughout the paper, first by using the definition of q-Al-Oboudi derivative operator and the concept of subordinating, we defined new class. After that, we used the new operator to introduce the new class $S^{n*}_{q,\zeta,\vartheta}(\lambda)$ which generalized classes of univalent functions. Finally, we obtained convolution properties results and coefficient estimates for this class and its subclasses. Our results generalize previous results.

5. Future studies

The authors suggest studying inclusion results of the defined class.

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Authors contributions

All the authors participated in constructing a new approach for solving the suggested problem and proving many theorems. All the authors approved the final manuscript.

Conflicts of interest

The authors declare that they have no competing interests.

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References

[1] Aouf MK, Mostafa AO, Elmorsy RE. Certain subclasses of analytic functions with varying arguments associated with q-difference operator. Afr Mat 2021;32:621–30.
[2] Al-Oboudi FM. On univalent functions defined by a generalized Sălăgean operator. Internat J Math Sci 2004;27: 1429–36.
[3] Annaby MH, Mansour ZS. q-Fractional calculus equations. Lecture Notes in Mathematises, vol 2056. Berlin: Springer; 2012.
[4] Bulboacă T. Differential subordinations and superordinations. Recent results. Cluj-Napoca: House of Scientific book Publ; 2005.
[5] Cotirla LI, Murugusundaramoorthy G. Starlike function based on Rusheweyh q-differential operator defined in Janowski Domain. Fractal Fract 2023;7(2):148. 16 pages.
[6] Jackson FH. On q-functions and a certain difference operator. Trans R Soc Edinb 1908;46:253–81.
[7] Miller SS, Mocanu PT. Subordinations of differential superordinations, complex variables. 2003. 48: no. 10. 815-825. 2023; 13, n4. 1491-1502.
[8] Murugusundaramoorthy G, Vijaya K, Breaz D, Cotirla IL. Subclasses of noshiro-type starlike harmonic functions involving q-srivastava–attiya operator. Math 2023;11: 4711.
[9] Sălăgean GS. Subclasses of univalent functions, complex analysis-fifth Romanian-Finnish seminar. Part 1 (Bucharest, 1981). Lecture Notes in Math. 1013. Berlin: Springer; 1983. p. 362–72.
[10] Seoudy TM, Aouf MK. Convolution properties for certain classes of analytic functions defined by q-derivative operator. Hind Publi Corp Appli Anal 2014;2014:1–7.
[11] Srivastava MH. Operators of basic(or q-) calculus and fractional q-calculus and their applications ingeometric function theory of complex analysis. Iran J Sci Technol Trans A Sci 2020;44:327–44.