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ORIGINAL ARTICLE

Approximating Special Monogenic Functions in Clifford Analysis

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Abstract

This paper deals with the approximation of specific classes of special monogenic function using exponential-derived and integral bases in Clifford analysis. To underscore the superiority of our results we provide illustrative examples and applications. These findings results extend existing knowledge from complex and quaternion forms to the context of Clifford analysis 2000 Mathematics Subject Classification: 30G35, 41A10.

Keywords: $T\rho$-property, Clifford algebra, Effectiveness, Exponential operators, Order, Type

1. Introduction

The theory of basic sets of polynomials (BPs) represents a crucial role in numerous branches of mathematics. It plays an essential role in many theoretical and practical areas such as partial differential equations, nonlinear analysis, mathematical physics, approximation theory, and mathematical modeling. The scientific basis of the BPs theory was established in the 20^{th} century [1], see also [2–4]. It says that if $f(z)$ is any analytic function, then $f(z)$ has an approximation value by using a base of $\{P_n(z)\}$ as $f(z) \sim \sum_n a_n P_n(z)$. It should be noted that the basic series is a generalization of a Taylor series, where $P_n(z)$ can be Chebyshe, Lagendre, Hermite, Laguerre, Bessel, Euler, and Bernoulli polynomials [5–8].

In [1,2], the authors initiated the study of BPs by considering regions of open and closed disks in one complex variable. The BPs have been generalized and extended in many directions [9–12]. One of the extensions is due to Abul-Ez and Constales, Abul-Ez, and Abul-Ez [12–14] and Abul-Ez *et al*. [9].

They presented some results in several domains (hyperballs, open hyperballs, open balls containing closed balls, origin, whole space) in Clifford analysis. Another direction of extensions refers to Malonek [15], Kishka *et al*. [10], El-sayed [11], Kumuyi and Nassif [16], El-sayed and Kishka [17], they employed appropriate functions in several complex variables in polycylinderical, hyperspherical and hyperelliptical regions. So (BPs) is explored in two directions: Several Complex Variables (S.C.V.) and Clifford Analysis. Through S.C.V., the theory of BPs are generalized from plane to spaces of the even dimensions ($\mathbb{C}^n \sim \mathbb{R}^{2n}, n \in \mathbb{N}$). In Clifford analysis, BPs extend from plane to spaces of both even and odd dimensions, as in $\mathcal{A}_m$ ($\mathbb{R}^{m+1}$, m $\in \mathbb{N}$).

In a recent paper, Hassan *et al*. [18] established a new approach of BPs of special monogenic polynomials in Fr'echet modules. The derivative of a complex function is a multifaceted analytical approach having topological and algebraic aspects. The derivative of BPs in one complex variable (resp., several variables) that are defined for open disks and closed disks (resp., polycylinderical, hyperspherical, and hyperelliptical regions) can be found in references [17,19,20] (resp., [10,11,16,17,21]). Recently, the framework of hypercomplex derivative bases of special monogenic polynomials in Clifford analysis has emerged as a very powerful

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and important tool in the theory of Clifford algebras [22,23].

The purpose of this paper is to introduce two different classes of differential and integral operators called exponential derived and integral bases. These operators will be used to deal with special monogenic polynomials in Clifford analysis. We deduce the topological properties of such operators such as convergence, effectiveness, T_ρ-property, order and type. Finally, we provide some examples and applications. The results here extend and improve the corresponding results announced by [24].

2. Preliminaries

Let R be the set of real numbers. Let E (resp., O) denote the set of even (resp., odd) integers.

The real Clifford analysis $\mathcal{A}_m$ is a 2^m -dimensional algebra with unit defined as follows: Given the orthogonal basis $e_0, e_1, \ldots, e_m$ of the linear space $\mathcal{A}_m$ (in other words $\mathbb{R}^{m+1}$). The basis is determined by $e_ie_j + e_je_i = -2\delta_{ij}$ for $i,j = 1,\ldots,m$. Let Ω be an open set of $\mathcal{A}_m$. If $x\in\mathcal{A}_m$, then $x:=x_0 + \sum_{i=1}^{m} e_ix_i$, where $x_i\in$ R and $e_0 = 1$. The conjugate of x is $\bar{x}$, where $\bar{x}:= x_0 - \sum_{i=1}^{m} e_ix_i$ as $\bar{e}_i = -e_i$ for $i = 1,\ldots,m$ and $e_0 = \bar{e}_0 = 1$. $|x|$ is the norm of x and is defined by $|x| = \sqrt{\sum_{i=0}^{m} x_i^2} = \sqrt{x\bar{x}} = \sqrt{\bar{x}x}$. The inverse of x is x^{-1} and is equal to $\frac{\bar{x}}{|x|^2}$ ($\forall$ $x\in\mathcal{A}_m$ and $x\neq 0$). If $,y\in\mathcal{A}_m$, then $|xy| \le 2^m|x|\,|y|$. In $\mathcal{A}_m$ the generalized Cauchy-Riemann operator is defined by $D = \sum_{i=0}^{m} e_i\frac{\partial}{\partial x_i}$ (for more details, see [15,25–27]).

Definition 2.1. Let $f^{(m)}: \Omega\to\mathcal{A}_m$, $m\ge 0$ be an $\mathcal{A}_m$-valued function. Then $f^{(m)}$ is called monogenic If $Df^{(m)} = f^{(m)}D = 0$ in Ω.

Definition 2.2. A polynomial $P^{(m)}(x)$ is called special monogenic (SMP) iff $DP^{(m)}(x) = 0$ and there is $a_{i,j}\in\mathcal{A}_m$ such that $P^{(m)}(x) = \sum_{i,j}^{\alpha}\bar{x}^i x^j a_{i,j}$, $\alpha<\infty$.

Definition 2.3. The function $f^{(m)}$ is called special monogenic in Ω if the following axioms hold:

(1) Ω is connected and open subset of A_m containing zero.
(2) The generalization form of the Taylor series of $f^{(m)}$ near zero (it must exist) is given by $f^{(m)}(x) = \sum_{n=0}^{\infty} P_n^{(m)}(x)$, where $P_n^{(m)}(x)$ is SMP.

Definition 2.4. A homogeneous SMP $P_n^{(m)}(x)$, $x\in\mathcal{A}_m$ is a function of the form $P_n^{(m)}(x) = p_n^{(m)}(x)\,\alpha_n$, where $\alpha_n\in\mathcal{A}_m$ and $p_n^{(m)}(x)$ is given by

$$p_n^{(m)}(x) = \frac{n!}{(m)_n}\sum_{k+l=n}\frac{\left(\frac{m-1}{2}\right)_k}{k!}\frac{\left(\frac{m+1}{2}\right)_l}{l!}\bar{x}^k x^l \tag{2.1}$$

where $(b)_l = b(b+1)\ldots(b+l-1)$ and $x\in A_m$, $b\in$ R.

Remark 2.1. The SMP sequence $\{P_n^{(m)}(x)\}$ from an Appell sequence if it satisfies that $\frac{\partial}{\partial x_0}p_n^{(m)}(x) = np_{n-1}^{(m)}(x)$.
For further information on monogenic functions, one can refer to [13,22,27–29].

Definition 2.5. A unitary left module X over $\mathcal{A}_m$ *is an abelian group* (X,t) with a mapping $\mathcal{A}_m\times X\to X$; $(\lambda, f^{(m)})\to\lambda f^{(m)}$ such that for all $\lambda,\mu\in X$ and $f^{(m)}, r^{(m)}\in\mathcal{A}_m$ such that the following axioms hold:

(1) $(\lambda + \mu)f^{(m)} = \lambda f^{(m)} + \mu f^{(m)}$,
(2) $(\lambda\mu)f^{(m)} = \lambda(\mu f^{(m)})$,
(3) $\lambda(f^{(m)} + r^{(m)}) = \lambda f^{(m)} + \lambda r^{(m)}$,
(4) $e_0 f^{(m)} = f^{(m)}$.

A unitary right module can be defined analogously as follows:

Definition 2.6. A unitary right module X over $\mathcal{A}_m$ *is an abelian group* (X,t) with a mapping $X\times\mathcal{A}_m\to X$; $(f^{(m)},\lambda)\to f^{(m)}\lambda$ such that for all $\lambda,\mu\in X$ and $f^{(m)}, r^{(m)}\in\mathcal{A}_m$ such that the following axioms hold:

(1) $f^{(m)}(\lambda + \mu) = f^{(m)}\lambda + f^{(m)}\mu$,
(2) $f^{(m)}(\lambda\mu) = (f^{(m)}\lambda)\mu$,
(3) $(f^{(m)} + r^{(m)})\lambda = f^{(m)}\lambda + r^{(m)}\lambda$,
(4) $f^{(m)}e_0 = f^{(m)}$.

It should be noted that if X is a field, then a unitary right (resp. left) module is a linear space and if $\mathbb{R}e_0 = \mathcal{A}_0\subset\mathcal{A}_m$, then X becomes a real linear space. In the sequel, all modules will be unitary right modules.
Let $X_{\mathbb{R}}$ be a unitary right module over $\mathcal{A}_m$, A function $\|.\|: X_{\mathbb{R}}\to[0,\infty)$, is called a seminorm on $X_{\mathbb{R}}$, if it fulfills for all $f^{(m)}, r^{(m)}\in X_{\mathbb{R}}, \lambda\in\mathcal{A}_m$ and $k\in\mathbb{R}$:

(1) $\|f^{(m)} + r^{(m)}\| \le \|f^{(m)}\| + \|r^{(m)}\|$,

(2) $\|f^{(m)}\| = 0 \Rightarrow f^{(m)} = 0$,
(3) there exists $c \geq 1$ such that $\|f^{(m)}\lambda\| \leq c|\lambda|\|f^{(m)}\|$ and $\|f^{(m)}k\| \leq c|k|\|f^{(m)}\|$.

If P is a family of seminorms on $X_{\mathbb{R}}$, then a subset Q of P is said to be a base of seminorms for P, if for each $p \in P$ there is a $q \in Q$ and a positive real s such that $p \leq sq$.

Definition 2.7. Let P be a family of countable proper system of seminorms on $X_{\mathbb{R}}$.

(1) If for any finite number $p_1, p_2, \ldots, p_l \in P$, with $l > 0$, there are $p \in P$ and $C > 0$ such that, for all $f^{(m)} \in X_{\mathbb{R}}$, $sup_{k=1,\ldots,l} P_k(f^{(m)}) \leq C_p(f^{(m)})$, then P is called a proper system of seminorms on $X_{\mathbb{R}}$.
(2) If $(X_{\mathbb{R}}, P)$ is a seminormed Hausdorff topological space such that $j < l$ implies that $P_j(f^{(m)}) \leq P_k(f^{(m)})$ for all $f^{(m)} \in X_{\mathbb{R}}$ and for all $f^{(m)} \in U \subseteq X_{\mathbb{R}}$, there are $\epsilon > 0$ and $M > 0$ such that $\{g^{(m)} \in X_{\mathbb{R}} : P_j(f^{(m)} - r^{(m)}) \leq \epsilon\} \subset U$ for all $j \leq M$ and $X_{\mathbb{R}}$ is complete with respect to the metric topology, then $X_{\mathbb{R}}$ is called Fréchet module.
(3) A sequence $(P_n)_{n \geq 0}$ in a Fréchet module $X_{\mathbb{R}}$ is said to converge to an element $f^{(m)} \in X_{\mathbb{R}}$ if and only if, for all $P_j \in P$, we have $\lim_{n \to \infty} P_j(f_n^{(m)} - f^{(m)}) = 0$.

Definition 2.8. A Banach module over $X_{\mathbb{R}}$ is an $X_{\mathbb{R}}$ over $\mathcal{A}_m$ equipped with a function $\|.\|_{\mathcal{A}_m} : \mathcal{A}_m \to [0, \infty)$ such that for all $f^{(m)}, r^{(m)} \in \mathcal{A}_m$ and $\lambda \in X_{\mathbb{R}}$:

(1) $\|0\|_{\mathcal{A}_m} = 0$,
(2) $\|f^{(m)} + r^{(m)}\|_{\mathcal{A}_m} \leq \|f^{(m)}\|_{\mathcal{A}_m} + \|r^{(m)}\|_{\mathcal{A}_m}$,
(3) there exists $c > 0$ such that $\|f^{(m)}\lambda\|_{\mathcal{A}_m} \leq c|\lambda|\|f^{(m)}\|_{\mathcal{A}_m}$,
(4) $\|f^{(m)}\|_{\mathcal{A}_m} = 0 \Rightarrow f^{(m)} = 0_{\mathcal{A}_m}$,
(5) $\mathcal{A}_m$ is complete with respect to the metric $d(f^{(m)}, r^{(m)}) = \|f^{(m)} - r^{(m)}\|_{\mathcal{A}_m}$.

It should be noted that a Banach module is a Fréchet module over $\mathcal{A}_m$.
Given an open ball $S(R)$ and a closed ball $\overline{S}(R)$ with both having a radius $R > 0$ and given an open ball $S^+(R)$ enclosing the closed ball $\overline{S}(R)$. We then consider: $\mathcal{T}[S(R)] = \{r^{(m)} \in \mathcal{T}[S(R)] : r^{(m)}(x)$ is SMF $\forall x \in S(R)\}$; $\mathcal{T}[\overline{S}(R)] = \{r^{(m)} \in \mathcal{T}[\overline{S}(R)] : r^{(m)}(x)$ is SMF $\forall x \in \overline{S}(R)\}$; $\mathcal{T}[S^+(R)] = \{r^{(m)} \in \mathcal{T}[S^+(R)] : r^{(m)}(x)$ is SMF $\forall x \in S^+(R)\}$; $\mathcal{T}[\infty] = r^{(m)} \in \mathcal{T}[\infty] : r^{(m)}(x)$ is an entire SMF in $\mathcal{A}_m\}$; $\mathcal{T}[0^+] = \{r^{(m)} \in \mathcal{T}[0^+] : r^{(m)}(x)$ is SMF at the origin$\}$. Then the countable family of a proper system of seminorms (which defines a *Fréchet* module) of each of the sets mentioned above are given respectively by: $\|r^{(m)}\|_{R^*} = sup_{\overline{S}(R^*)}|r^{(m)}(x)|$, $x \in \mathcal{A}_m\, \forall R^* < R$, $r^{(m)} \in \mathcal{T}[S(R)]$; $\|r^{(m)}\|_R = sup_{\overline{S}(R)}|r^{(m)}(x)|$, $x \in \mathcal{A}_m$, $r^{(m)} \in \mathcal{T}[\overline{S}(R)]$; $\|r^{(m)}\|_{R^*} = sup_{\overline{S}(R^*)}|r^{(m)}(x)|$, $x \in \mathcal{A}_m\, \forall R < R^*$, $r^{(m)} \in \mathcal{T}[S^+(R)]$; $\|r^{(m)}\|_n = sup_{\overline{S}(n)}|r^{(m)}(x)|$, $x \in \mathcal{A}_m$, $n < \infty$, $\forall r^{(m)} \in \mathcal{T}[\infty]$; $\|r^{(m)}\|_\epsilon = sup_{\overline{S}(\epsilon)}|r^{(m)}(x)|$, $x \in \mathcal{A}_m, \epsilon > 0, \forall r^{(m)} \in \mathcal{T}[0^+]$ where $\overline{S}(\epsilon)$ is a closed ball of radius ϵ surrounding 0.

Definition 2.9. Let $\{P_n^{(m)}(x)\}$ be a sequence of Fréchet module $X_{\mathbb{R}}$. Then $P_n^{(m)}(x)$ is a base, if it can be expressed in the form

$$p_n^{(m)}(x) = \sum_k P_k^{(m)} \pi_{n,k}^{(m)}, \pi_{n,k}^{(m)} \in \mathcal{A}_m \tag{2.2}$$

Where

$$P_n^{(m)}(x) = \sum_k p_k^{(m)} P_{n,k}^{(m)}, P_{n,k}^{(m)} \in \mathcal{A}_m \tag{2.3}$$

$\Pi^{(m)} = (\pi_{n,k}^{(m)})$ and $P^{(m)} = (P_{n,k}^{(m)})$ are the Clifford matrices of operators and coefficients of the base $\{P_n^{(m)}(x)\}$ in $\mathcal{A}_m$.

Remark 2.2. Every simple base is a base of degree n $(n \in N)$ and every Cannon base (i.e., $\lim_{n \to \infty} sup\{N_n\}^{\frac{1}{n}} = 1$, where N_n is the number of non-zero terms $\pi_{n,k}^{(m)}$ in (2.2)) is also a base.

Theorem 2.1. [13] *The necessary and sufficient condition for a SMP to be a base in* $\overline{S}(R)$ *is* $P^{(m)}\Pi^{(m)} = \Pi^{(m)}P^{(m)} = I$, *where I is unit matrix.*

Remark 2.3. *For Appell SMP, the Cauchy's inequality in the neighborhood of* $\overline{S}(R)$, *can be written as*

$$\left|P_{n,i}^{(m)}\right| \leq \frac{\|P_n^{(m)}\|_R}{R^i} \tag{2.4}$$

If $r^{(m)}(x) = \sum_n p_n^{(m)}(x) a_n^{(m)}(x)$ *is a SMF on a Fréchet module* $X_{\mathbb{R}}$, *then one can write*

$$r^{(m)}(x) = \sum_k P_n^{(m)}(x) \Pi_n(r^{(m)}) \tag{2.5}$$

where

$$\Pi_n\left(r^{(m)}\right)=\sum\nolimits_k \pi_{n,k}^{(m)}a_k\left(r^{(m)}\right) \tag{2.6}$$

Definition 2.10. A base $\{P_n^{(m)}(x)\}$ *in a Fréchet module* $X_{\mathbb{R}}$ *is called effective, if the basic series* (2.5) *converges uniformly to* $r^{(m)}(x)$ *on* $X_{\mathbb{R}}$.
The Cannon function $\lambda_{P^{(m)}}(R)$ *concerning the effectiveness of the base in the Fréchet module* $X_{\mathbb{R}}$, *is given by*

$$\lambda_{P^{(m)}}(R)=\limsup_{n\to\infty}\left\{\omega_{P_n^{(m)}}(R)\right\}^{\frac{1}{n}} \tag{2.7}$$

Where

$$\omega_{P_n^{(m)}}(R)=\sum_k\left\|P_k^{(m)}\ \pi_{n,k}^{(m)}\right\|_R \tag{2.8}$$

$$\left\|P_n^{(m)}\right\|_R=\sup_{\overline{S}(R)}\left|P_n^{(m)}(x)\right| \tag{2.9}$$

Theorem 2.2. Let $\{P_n^{(m)}(x)\}$ *be a sequence of bases in Fréchet modules* $[\overline{S}(R)]$, $\mathcal{T}\ [S(R)]$, $\mathcal{T}\ [S(R^+)]$, $\mathcal{T}\ [\infty]$ *or* $\mathcal{T}\ [0^+]$. *Then* $\{P_n^{(m)}(x)\}$ *is effective if and only if* $\lambda_{P^{(m)}}(R)=R$, $\lambda_{P^{(m)}}(R^*)<R\ \forall R^*<R$, $\lambda_{P^{(m)}}(R^+)=R$, $\lambda_{P^{(m)}}(R)<\infty\,\forall R<\infty$, *or* $\lambda_{P^{(m)}}(0^+)=0$, *respectively.*

Definition 2.11. The order $\rho_{P^{(m)}}$ *and type* $\tau_{P^{(m)}}$ *of a base* $\{P_n^{(m)}(x)\}$ *of SMP are defined as follows*:

$$\rho_{P^{(m)}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\omega_{P_n^{(m)}}(R)}{n\log n} \tag{2.10}$$

$$\tau_{P^{(m)}}=\lim_{R\to\infty}\frac{e}{\rho}\limsup_{n\to\infty}\frac{\left\{\omega_{P_n^{(m)}}(R)\right\}^{\frac{1}{n\rho}}}{n} \tag{2.11}$$

If a base $\{P_n^{(m)}(x)\}$ *is of order* $\rho_{P^{(m)}}$ (*resp., type* $\tau_{P^{(m)}}$), *then it will represent in any* $\overline{S}(R)$ *every entire SMF of order* $\frac{1}{\rho_{P^{(m)}}}$ (*resp., type* $\frac{1}{\tau_{P^{(m)}}}$).

Definition 2.12. The T_ρ *– property of SMP in* $\mathcal{T}[\overline{S}(R)]$, $\mathcal{T}\ [S(R)]$ *or at the origin, where* $0<\rho<\infty$, *means that the SMP represents all entire SMFs of order less than* ρ *in the same domain.*

Theorem 2.3. *A* SMP *has* $T\rho$*-property in* $\mathcal{T}\ [\overline{S}(R)]$, $\mathcal{T}\ [S(R)]$ or at the origin iff $\omega_{P^{(m)}}(R)\le\frac{1}{\rho}$, $\omega_{P^{(m)}}(R^*)\le\frac{1}{\rho}\ \forall R^*<R$ *or* $\omega_{P^{(m)}}(0^+)\le\frac{1}{\rho}$, *respectively, where*

$$\omega_{P^{(m)}}(R)=\lim_{n\to\infty} sup\frac{log\ \omega_{P_n^{(m)}}(R)}{n\ log\ n} \tag{2.12}$$

For more properties, one can see [13,27].

3. Exponential operators on SMPs

We now, state the new classes of exponential derivative-type and integral-type operators in A_m.

Definition 3.1. Let $\{P_n^{(m)}(x)\}$ be a base. Then:

(1) the exponential derivative operator EXP(θ), is defined by

$$EXP(\theta)p_n^{(m)}(x)=e^n p_n^{(m)}(x) \tag{3.1}$$

(2) (ii) the exponential integral operator EXP(ξ), is defined by

$$EXP(\xi)p_n^{(m)}(x)=e^{\frac{1}{n+1}}p_n^{(m)}(x) \tag{3.2}$$

Now, for $\theta^j=\theta\theta^{j-1}, j\in N$, we introduce a θ-operator as $\frac{n}{n+1}\frac{\partial}{\partial x_0}p_{n+1}^{(m)}(x)=\theta p_n^{(m)}(x)$ and for $\xi^i=\xi\xi^{i-1}, i\in N$, we introduce a ξ -operator as $\frac{n}{n+1}I_0p_{n-1}^{(m)}(x)=\xi p_n^{(m)}(x)$.
Then from (2.3), we have

$$EXP(\theta)P_n^{(m)}(x)=\sum_k e^k p_k^{(m)}(x)\ P_{n,k}^{(m)} \tag{3.3}$$

$$EXP(\xi)P_n^{(m)}(x)=\sum_k e^{e^{\frac{1}{k+1}}}p_k^{(m)}(x)\ P_{n,k}^{(m)} \tag{3.4}$$

In the sequel, we shall write $\{E_n^{(m,\theta)}(x)\}$ (resp., $\{E_n^{(m,\xi)}(x)\}$) abbreviation of $\{EXP(\theta)P_n^{(m)}(x)\}$ (resp., $\{EXP(\xi)P_n^{(m)}(x)\}$).
Now, we establish the following theorem:

Theorem 3.1. Let $\{P_n^{(m)}(x)\}$ *be a base of SMPs in* $\mathcal{A}_m$. *Then* $\{E_n^{(m,\theta)}(x)\}$ *is also a base.*

Proof. *Since every matrix of coefficients* (*resp., operators*) *of* $\{E_n^{(m,\theta)}(x)\}$ *can be expressed as*

$$E^{(m,\theta)}=\left(E_{n,k}^{(m,\theta)}\right)=\left(e^kP_{n,k}^{(m)}\right)\left(\text{resp.,}\Pi^{(m,\theta)}=\left(\pi_{n,k}^{(m,\theta)}\right)=\left(\frac{1}{e^n}\pi_{n,k}^{(m)}\right)\right),\text{it follows that}$$

$$E^{(m,\theta)}\Pi^{(m,\theta)}=\left(\sum_k E_{n,k}^{(m,\theta)}\pi_{k,n}^{(m,\theta)}\right)=\left(\sum_k P_{n,k}^{(m)}\pi_{k,n}^{(m)}\right)=I.$$

Similarly

$$\Pi^{(m,\theta)}E^{(m,\theta)}=\left(\sum_k \pi_{n,k}^{(m,\theta)}E_{k,n}^{(m,\theta)}\right)=\left(\frac{e^k}{e^n}\sum_k \pi_{n,k}^{(m)}P_{k,n}^{(m)}\right)=I.$$

Hence, by Theorem 2.1, *we conclude that* $\{E_n^{(m,\theta)}(x)\}$ *is also a base.*
In a similar way, one can show that $\{E_n^{(m,\xi)}(x)\}$ *is also a base, whenever* $\{P_n^{(m)}(x)\}$ *is a base in* A_m.

4. Effectiveness of $E_n^{(m,\theta)}(x)$ and $E_n^{(m,\xi)}(x)$

In this section, we state some results concerning to the effectiveness in Fréchet modules.

Theorem 4.1. *The* $\{E_n^{(m,\theta)}(x)\}$ *is effective for* $\mathcal{T}\ [\overline{S}(R)]$, *if the following conditions hold*:

(1) *the base* $\{P_n^{(m)}(x)\}$ *of SMP is effective for* $\mathcal{T}\ [\overline{S}(R)]$.
(2) $\lim_{n\to\infty}\frac{D_n}{n}=1$, *where* D_n *is the highest degree of* (2.2).

Proof. *Since* $\{P_n^{(m)}(x)\}$ *is a base, it follows that* $\| P_n^{(m)} \|_R = \sup_{\overline{S}(R)}\left|P_n^{(m)}(x)\right|$ *and* $\| E_n^{(m,\theta)} \|_R = \sup_{\overline{S}(R)}\left|E_n^{(m,\theta)}(x)\right|$.
Simplifying, we have

$$\| E_n^{(m,\theta)}\pi_{n,k}^{(m,\theta)} \|_R=\sup_{\overline{S}(R)}\left|E_n^{(m,\theta)}(x)\pi_{n,k}^{(m,\theta)}\right|=\sup_{\overline{S}(R)}\left|\left(\sum_j e^j p_j^{(m)}P_{k,j}^{(m)}\right)\frac{\pi_{n,k}^{(m)}}{e^n}\right|$$
$$\leq 2^m\left\|P_n^{(m)}\pi_{n,k}^{(m)}\right\|_R\sum_j e^j=2^m\left\|P_n^{(m)}\pi_{n,k}^{(m)}\right\|_R e^{d_k}(d_k+1) \tag{4.1}$$

where the degree d_k *of* $P_k^{(m)}(x)$, *should be less than or equal to* D_k.
Using (2.8) and (2.7) *we get*

$$\omega_{E_n^{(m,\theta)}}(R)=\sum_k\left\|E_k^{(m,\theta)}\pi_{n,k}^{(m,\theta)}\right\|_R\leq\frac{1}{e^n}\sum_k\left\|P_k^{(m)}\pi_{n,k}^{(m)}\right\|_R e^{d_k}(d_k+1)\leq 2^m\frac{1}{e^n}e^{d_n}(d_n+1)\omega_{P_n^{(m)}}(R)\leq 2^m e^{D_n-n}(D_n+1)\omega_{P_n^{(m)}}(R) \tag{4.2}$$

This implies that $\lambda_{E^{(m,\theta)}}\leq\lambda_{P^{(m)}}\leq R$. *This, together with the fact* $\lambda_{E^{(m,\theta)}}\geq R$, *implies that* $\lambda_{E^{(m,\theta)}}=R$ *and the desired conclusion is reached.*

Remark 4.1. Theorem 4.1 *is still true, if we replace* $\{E_n^{(m,\theta)}(x)\}$ *by* $\{E_n^{(m,\xi)}(x)\}$.

Theorem 4.2. Both $\{E_n^{(m,\theta)}(x)\}$ *and* $\{E_n^{(m,\xi)}(x)\}$ *are effective for* $\mathcal{T}\ [S(R)], \mathcal{T}\ [S(R^+)], \mathcal{T}\ [0^+]$ *and* $\mathcal{T}\ [\infty]$ *spaces, if the following conditions are satisfied*:

(1) *the base* $\{P_n^{(m)}(x)\}$ *of SMP is effective for* $\mathcal{T}\ [S(R)], \mathcal{T}\ [S(R^+)], \mathcal{T}\ [0^+]$ *and* $\mathcal{T}\ [\infty]$, *respectively,*
(2) $\lim_{n\to\infty}\frac{D_n}{n}=1$, *where* D_n *is the highest degree of* (2.2).

The proof *of* Theorem 4.2 *is similar to* [30] *and* Theorem 4.1, *so we omit the details.*
We give an example to show that the hypothesis (*ii*) *in* Theorem 4.1 *is necessary.*

Example 4.1. *Suppose* $\{P_n^{(m)}(x)\}$ *is a base of SMP such that*

$$P_n^{(m)}(x)=\begin{cases} p_n^{(m)}(x), n\in\mathbb{E},\\ p_n^{(m)}(x)+p_l^{(m)}(x), l=2n, n\in\mathbb{O}.\end{cases}$$

If $n\in\mathbb{E}$, *then*

$$p_n^{(m)}(x)=P_n^{(m)}(x)\Rightarrow\omega_{P_n^{(m)}}(R)=R^n\Rightarrow\omega_{P_n^{(m)}}(1)=1,$$
$$\text{when } R=1\Rightarrow\lim_{n\to\infty}\left\{\omega_{P_{2n}^{(m)}}(1)\right\}^{\frac{1}{2n}}=1$$

If $n\in\mathbb{O}$, *then*

$$p_n^{(m)}(x)=P_n^{(m)}(x)-P_l^{(m)}(x)\Rightarrow\omega_{P_n^{(m)}}(R)=R^n+2R^l\Rightarrow\omega_{P_n^{(m)}}(1)=3, \text{when}$$
$$R=1\Rightarrow\lim_{n\to\infty}\left\{\omega_{P_{2n+1}^{(m)}}(1)\right\}^{\frac{1}{2n+1}}=1$$

Hence $\{P_n^{(m)}(x)\}$ *is effective for* $\mathcal{T}\ [\overline{S}(1)]$.
Construct the set $\{E_n^{(m,\theta)}(x)\}$ *as follows*:

$$E_n^{(m,\theta)}(x)=\begin{cases} e^n p_n^{(m)}(x), n\in\mathbb{E},\\ e^n p_n^{(m)}(x)+e^l p_l^{(m)}(x), l=2n, n\in\mathbb{O}.\end{cases}$$

If $n\in\mathbb{E}$, *then*

$$p_n^{(m)}(x)=\frac{1}{e^n}E_n^{(m,\theta)}(x)\Rightarrow\omega_{E_n^{(m,\theta)}}(R)=R^n\Rightarrow\omega_{E_n^{(m,\theta)}}(1)=1,$$
$$\text{when } R=1\Rightarrow\lim_{n\to\infty}\left\{\omega_{E_{2n}^{(m,\theta)}}(1)\right\}^{\frac{1}{2n}}=1$$

If $n\in\mathbb{O}$, *then*

$$p_n^{(m)}(x)=\frac{1}{e^n}\left[E_n^{(m,\theta)}(x)-E_l^{(m,\theta)}(x)\right]\Rightarrow\omega_{E_n^{(m,\theta)}}(R)=R^n+2e^nR^l\Rightarrow\omega_{E_n^{(m,\theta)}}(1)=1+2e^n\Rightarrow\lim_{n\to\infty}\left\{\omega_{E_{2n+1}^{(m,\theta)}}(1)\right\}^{\frac{1}{2n+1}}=e.$$

Hence $\{E_n^{(m,\theta)}(x)\}$ *is not effective for* $\mathcal{T}\ [\overline{S}(1)]$.

As an application of our results, we consider the following bases $\{\mathbb{P}_n^{(m)}(x)\}$ *and* $\{\mathbb{Q}_n^{(m)}(x)\}$ *of the Bessel and generalized Bessel polynomials in* $\mathcal{T}\,[\overline{S}(R)]$, *respectively*:

$$\mathbb{P}_0^{(m)}(x)=1,\mathbb{P}_n^{(m)}(x)=\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!2^k}p_k^{(m)}(x);\ (n\geq 1),$$

$$\mathbb{Q}_0^{(m)}(x)=1,\mathbb{Q}_n^{(m)}(x)=1+\sum_{k=0}^{n}\frac{n!(n+b-1)(n+b)\ldots(n+k+b-2)}{k!(n-k)!a^k}\times p_k^{(m)}(x).$$

According to [6,7], *we have that both* $\{\mathbb{P}_n^{(m)}(x)\}$ *and* $\{\mathbb{Q}_n^{(m)}(x)\}$ *are effective for* $\mathcal{T}\,[\overline{S}(R)]$ *and condition* (*ii*) *of* Theorem 4.1 *holds. Then as an immediate application of* Theorem 4.1 *and Remark*

4.1, *we conclude that* $\{\mathbb{P}_n^{(m,\theta)}(x)\}$, $\{\mathbb{P}_n^{(m,\xi)}(x)\}$, $\{\mathbb{Q}_n^{(m,\theta)}(x)\}$ *and* $\{\mathbb{Q}_n^{(m,\xi)}(x)\}$ *are also effective for* $\mathcal{T}\,[\overline{S}(R)]$.

5. Computations of order, type and T_ρ-property

In this section, we study the relationship between $\rho_{P^{(m)}}$, $\tau_{P^{(m)}}$ of $\{P_n^{(m)}(x)\}$ and $\rho_{E^{(m,\theta)}}$, $\tau_{E^{(m,\theta)}}$ of $\{E_n^{(m,\theta)}(x)\}$.

Theorem 5.1. *If* $\{P_n^{(m)}(x)\}$ *is a SMPs of order* $\rho_{P^{(m)}}$ *and type* $\tau_{P^{(m)}}$ *such that*

$$D_n=O[n] \tag{5.1}$$

Then the following statement holds: $\rho_{E^{(m,\theta)}}\leq\rho_{P^{(m)}}$ *and* $\tau_{E^{(m,\theta)}}\leq\tau_{P^{(m)}}$, *whenever* $\rho_{E^{(m,\theta)}}=\rho_{P^{(m)}}$, *where* $\rho_{E^{(m,\theta)}}$ *is the order* (*resp., type*) *of* $\{E_n^{(m,\theta)}(x)\}$.

Proof. *It follows from* (2.10) and (4.2), *that:*

$$\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\omega_{E_n^{(m,\theta)}}(R)}{n\log n}$$
$$\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log 2^m e^{D_n-n}(D_n+1)+\log\omega_{P_n^{(m)}}(R)}{n\log n}.$$

This implies that $\rho_{E^{(m,\theta)}}\leq\rho_{P^{(m)}}$.
From $\rho_{E^{(m,\theta)}}=\rho_{P^{(m)}}$ *and* (2.11), *we have*

$$\lim_{R\to\infty}\frac{e}{\rho_{E^{(m,\theta)}}}\limsup_{n\to\infty}\frac{\left\{\omega_{E_n^{(m,\theta)}}(R)\right\}^{\frac{1}{n\left(\rho_{E^{(m,\theta)}}\right)}}}{n}\leq\lim_{R\to\infty}\frac{e}{\rho_{P^{(m)}}}\limsup_{n\to\infty}\frac{\left\{\omega_{P_n^{(m)}}(R)\right\}^{\frac{1}{n\left(\rho_{P^{(m)}}\right)}}}{n},$$

which gives $\tau_{E^{(m,\theta)}}\leq\tau_{P^{(m)}}$. *Thus the theorem is proved. Giving an example which supports* Theorem 5.1:

Example 5.1. Suppose $\{P_n^{(m)}(x)\}$ *is a base of SMP such that* $P_0^{(m)}=1$, $P_n^{(m)}=n^n+p_n^{(m)}(x)$.
Then

$$\omega_{P_n^{(m)}}(R)=2n^n+R^n,$$

$$\rho_{P^{(m)}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log(2n^n+R^n)}{n\log n}=1,$$

$$\tau_{P^{(m)}}=\lim_{R\to\infty}\frac{e}{1}\limsup_{n\to\infty}\frac{\{2n^n+R^n\}^{\frac{1}{n}}}{n}=e.$$

Constructing the corresponding $\{E_n^{(m,\theta)}(x)\}$ *base as following*: $E_0^{(m,\theta)}=1$, $E_n^{(m,\theta)}=n^n+e^np_n^{(m)}(x)$.
Then

$$\omega_{E_n^{(m,\theta)}}(R)=\frac{2n^n}{e^n}+R^n,$$

So that

$$\rho_{E^{(m,\theta)}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\left(\frac{2n^n}{e^n}+R^n\right)}{n\log n}=1,$$

$$\tau_{E^{(m,\theta)}}=\lim_{R\to\infty}\frac{e}{1}\limsup_{n\to\infty}\frac{\left\{\frac{2n^n}{e^n}+R^n\right\}^{\frac{1}{n}}}{n}=e.$$

Hence the two bases $\{P_n^{(m)}(x)\}$ *and* $\{E_n^{(m,\theta)}(x)\}$ *have the same order* 1 *and type* e *in* $\mathcal{T}[\overline{S}(R)]$.
Next example shows the importance of condition (5.1).

Example 5.2. Suppose $\{P_n^{(m)}(x)\}$ *is a base of SMP in such that*

$$P_n^{(m)}(x)=\begin{cases}p_n^{(m)}(x), n\in\mathbb{E},\\ p_n^{(m)}(x)+\frac{v}{b^{2v}}p_{2v}^{(m)}(x), v=n^n, R=b\ n\in\mathbb{O}.\end{cases}$$

If $n\in\mathbb{O}$, *then*

$$p_n^{(m)}(x)=P_n^{(m)}(x)-\frac{v}{b^{2v}}P_{2v}^{(m)}(x)\Rightarrow\omega_{P_n^{(m)}}(R)$$
$$=R^n+2v\left(\frac{R}{b}\right)^{2v},\text{but } R=b\Rightarrow\omega_{P_n^{(m)}}(R)=R^n+2v$$
$$\Rightarrow\rho_{P^{(m)}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log(R^n+2v)}{n\log n}=1.$$

Hence $\{P_n^{(m)}(x)\}$ *is of order* 1 *in* $\mathcal{T}[\overline{S}(R)]$.
construct the set $\{E_n^{(m,\theta)}(x)\}$ *as follows*:

$$E_n^{(m,\theta)}(x)=\begin{cases} e^n p_n^{(m)}(x), n\in\mathbb{E}, \\ e^n p_n^{(m)}(x)+\frac{v}{b^{2v}}e^{2v}p_{2v}^{(m)}(x), v=n^n, R=b\, n\in\mathbb{O}. \end{cases}$$

If $n\in\mathbb{O}$, then

$$p_n^{(m)}(x)=\frac{1}{e^n}E_n^{(m,\theta)}(x)-\frac{v}{e^n b^{2v}}E_{2v}^{(m,\theta)}(x)$$

$$\Rightarrow \omega_{E_n^{(m,\theta)}}(R)=R^n+\frac{2v}{e^n}\left(\frac{eR}{b}\right)^{2v}, \text{but } R=b$$

$$\Rightarrow \omega_{E_n^{(m,\theta)}}(R)=R^n+\frac{2ve^{2v}}{e^n}$$

$$\Rightarrow \rho_{E^{(m,\theta)}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\left(R^n+\frac{2ve^{2v}}{e^n}\right)}{n\log n}=\infty.$$

Hence $\{E_n^{(m,\theta)}(x)\}$ doesn't achieve the inequality $\rho_{E^{(m,\theta)}}\le\rho_{P^{(m)}}$ in $\mathcal{T}[\overline{S}(R)]$.
The following theorem establishes the $T_\rho-$ property of $\{E_n^{(m,\theta)}(x)\}$ for $\mathcal{T}[\overline{S}(R)]$:

Theorem 5.2. The base $\{E_n^{(m,\theta)}(x)$ *has a $T_\rho-$ property in $\overline{S}(R)$ $(R\ge 0)$, if:*

(i) $\{P_n^{(m)}(x)\}$ has T_ρ-property in $\overline{S}(R)$,
(ii) $\lim_{n\to\infty}\frac{\log D_n}{n\log n}=0$.

Proof. Let $\omega_{E^{(m,\theta)}}$ be defined by

$$\omega_{E^{(m,\theta)}}(R)=\lim_{n\to\infty}\frac{\log\omega_{E_n^{(m,\theta)}}(R)}{n\log n} \tag{5.2}$$

Using (4.2), (2.12) and Theorem 5.2, we have

$$\omega_{E^{(m,\theta)}}(R)\le\omega_{P^{(m)}}(R)\le\frac{1}{\rho}.$$

Remark 5.1. *Theorems 5.1 and 5.2 remain valid if we replace $\{E_n^{(m,\theta)}(x)\}$ by $\{E_n^{(m,\xi)}(x)\}$ and have similar proofs.*
The following example shows that the condition (5.1) *in Theorem 5.1 is necessary.*

Example 5.3. Suppose $\{P_n^{(m)}(x)\}$ *is a base of SMP such that $P_0^{(m)}(x)=1$, $P_n^{(m)}(x)=p_n^{(m)}(x)+2^b p_b^{(m)}(x)$, where b is the nearest natural number to $2n\log n$.*
Then

$$\omega_{P_n^{(m)}}(R)=R^n+2^{b+1}R^b.$$

If $R=1$, then we have

$$\omega_{P_n^{(m)}}(1)=1+2^{b+1},$$

$$\omega_{P^{(m)}}(R)=\limsup_{n\to\infty}\frac{\log\omega_{P_n^{(m)}}(1)}{n\log n}=2\log 2$$

Hence $\{P_n^{(m)}(x)\}$ has $T_{\frac{1}{2\log 2}}$-property in $\mathcal{T}[\overline{S}(R)]$. Let $\{E_n^{(m,\theta)}(x)\}$ be a base constructed as $E_0^{(m,\theta)}(x)=1$, $E_n^{(m,\theta)}(x)=e^n p_n^{(m)}(x)+2^b e^b p_b^{(m)}(x)$.
Then

$$\omega_{E_n^{(m,\theta)}}(R)=R^n+2^{b+1}e^{b-n}R^b,$$

so that

$$\omega_{E^{(m,\theta)}}(R)=\limsup_{n\to\infty}\frac{\log\omega_{E_n^{(m,\theta)}}(1)}{n\log n}=2\log 2\, e$$

Hence $\{E_n^{(m,\theta)}(x)\}$ doesn't have $T_{\frac{1}{2\log 2}}$-property in $\overline{S}(R)$.
Finally, we illustrate the usefulness of Theorem 5.2 by giving some applications. In the field of Clifford analysis, Hassan and Aloui [5] *showed the Bernoulli special monogenic polynomials $\{\mathfrak{B}_n(x)\}$ is of order 1 and type $\frac{1}{2\pi}$ and the Euler special monogenic polynomials $\{\mathfrak{E}_n(x)\}$ is of order 1 and type $\frac{1}{\pi}$. Also both polynomials have property T_1.*
Now, using Theorem 5.2, we obtain the following results:

(1) *The exponential SMP of Bernoulli $\{\mathfrak{B}_n^{(m,\theta)}(x)\}$ is of order 1 and type $\frac{1}{2\pi}$.*
(2) *The exponential SMP of Euler $\{\mathfrak{E}_n^{(m,\theta)}(x)\}$ is of order 1 and type $\frac{1}{\pi}$.*
(3) *The exponential SMP of Bernoulli $\{\mathfrak{B}_n^{(m,\theta)}(x)\}$ has T_1-property.*
(4) *The exponential SMP of Euler $\{\mathfrak{E}_n^{(m,\theta)}(x)\}$ has T_1-property.*

5.1. Conclusions

In this paper, we have established a novel set of polynomial bases in $\mathcal{A}_m$ through the utilization of exponential derived and integral operators in Clifford analysis. The operators can be viewed as a generalization of the complex form **C** (when m = 1) and quaternion form **H** (when m = 2). We investigated the convergence properties of the effectiveness of operators $E_n^{(m,\theta)}(x)$ and $E_n^{(m,\xi)}(x)$, analyzing their order, type and T_ρ-property. Additionally, we provided illustrative examples and applications to elucidate the principal findings.

Conflicts of interest

There are no conflicts of interest.

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