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Approximating Special Monogenic Functions in Clifford Analysis

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Abstract

This paper deals with the approximation of specific classes of special monogenic function using exponential-derived and integral bases in Clifford analysis. To underscore the superiority of our results we provide illustrative examples and applications. These findings results extend existing knowledge from complex and quaternion forms to the context of Clifford analysis. 2000 Mathematics Subject Classification: 30G35, 41A10.

Keywords: Tr-property, Clifford algebra, Effectiveness, Exponential operators, Order, Type

1. Introduction

The theory of basic sets of polynomials (BPs) represents a crucial role in numerous branches of mathematics. It plays an essential role in many theoretical and practical areas such as partial differential equations, nonlinear analysis, mathematical physics, approximation theory, and mathematical modeling. The scientific basis of the BPs theory was established in the 20th century [1], see also [2–4]. It says that if \( f(z) \) is any analytic function, then \( f(z) \) has an approximation value by using a base of \( \{P_n(z)\} \) as \( f(z) \approx \sum_n a_n P_n(z) \). It should be noted that the basic series is a generalization of a Taylor series, where \( P_n(z) \) can be Chebyshe, Lagendre, Hermite, Laguerre, Bessel, Euler, and Bernoulli polynomials [5–8].

In [1,2], the authors initiated the study of BPs by considering regions of open and closed disks in one complex variable. The BPs have been generalized and extended in many directions [9–12]. One of the extensions is due to Abul-Ez and Constales, Abul-Ez, and Abul-Ez [12–14] and Abul-Ez et al. [9].

They presented some results in several domains (hyperballs, open hyperballs, open balls containing closed balls, origin, whole space) in Clifford analysis. Another direction of extensions refers to Malonek [15], Kishka et al. [10], El-sayed [11], Kumuij and Nassif [16], El-sayed and Kishka [17], they employed appropriate functions in several complex variables in polycylindrical, hyperspherical and hyperelliptical regions. So (BPs) is explored in two directions: Several Complex Variables (S.C.V.) and Clifford Analysis. Through S.C.V., the theory of BPs are generalized from plane to spaces of the even dimensions \( (\mathbb{C}^n \sim \mathbb{R}^{2n}, n \in \mathbb{N}) \). In Clifford analysis, BPs extend from plane to spaces of both even and odd dimensions, as in \( \mathcal{A}_m (\mathbb{R}^{m+1}, m \in \mathbb{N}) \).

In a recent paper, Hassan et al. [18] established a new approach of BPs of special monogenic polynomials in Fréchet modules. The derivative of a complex function is a multifaceted analytical approach having topological and algebraic aspects. The derivative of BPs in one complex variable (resp., several variables) that are defined for open disks and closed disks (resp., polycylindrical, hyperspherical, and hyperelliptical regions) can be found in references [17,19,20] (resp., [10,11,16,17,21]). Recently, the framework of hypercomplex derivative bases of special monogenic polynomials in Clifford analysis has emerged as a very powerful...
and important tool in the theory of Clifford algebras [22,23].

The purpose of this paper is to introduce two different classes of differential and integral operators called exponential derived and integral bases. These operators will be used to deal with special monogenic polynomials in Clifford analysis. We deduce the topological properties of such operators such as convergence, effectiveness, $T_p$-property, order and type. Finally, we provide some examples and applications. The results here extend and improve the corresponding results announced by [24].

2. Preliminaries

Let $R$ be the set of real numbers. Let $E$ (resp., $O$) denote the set of even (resp., odd) integers.

The real Clifford analysis $A_m$ is a $2^m$ -dimensional algebra with unit defined as follows: Given the orthogonal basis $e_0, e_1, \ldots, e_m$ of the linear space $A_m$ (in other words $R^{m+1}$). The basis is determined by $e_i e_j + e_j e_i = -2 \delta_{ij}$ for $i,j = 1, \ldots, m$. Let $\Omega$ be an open set of $A_m$. If $x \in \Omega$, then $x = x_0 + \sum_{i=1}^m e_i x_i$, where $x_i \in R$ and $e_0 = 1$. The conjugate of $x$ is $\bar{x}$, where $\bar{x} := x_0 - \sum_{i=1}^m e_i x_i$ as $e_i = -e_i$ for $i = 1, \ldots, m$ and $e_0 = \bar{e}_0 = 1$. $|x|$ is the norm of $x$ and is defined by $|x| = \sqrt{\sum_{i=0}^m x_i^2} = \sqrt{\bar{x} x} = \sqrt{x \bar{x}}$. The inverse of $x$ is $x^{-1}$ and is equal to $\frac{1}{x} \bar{x}$ for all $x \in A_m$ and $x \neq 0$. If $x, y \in A_m$, then $|xy| \leq 2^m |x| |y|$. In $A_m$ the generalized Cauchy-Riemann operator is defined by $D = \sum_{i=0}^m \frac{\partial}{\partial x_i}$ (for more details, see [15,25–27]).

Definition 2.1. Let $f^{(m)} : \Omega \to A_m$, $m \geq 0$ be an $A_m$-valued function. Then $f^{(m)}$ is called monogenic if $D f^{(m)} = f^{(m)} D = 0$ in $\Omega$.

Definition 2.2. A polynomial $P^{(m)}(x)$ is called special monogenic (SMP) if $D P^{(m)}(x) = 0$ and there is $a_{ij} \in A_m$ such that $P^{(m)}(x) = \sum_{i,j} a_{ij} x_i^j$, $\alpha < \infty$.

Definition 2.3. The function $f^{(m)}$ is called special monogenic in $\Omega$ if the following axioms hold:

1. $\Omega$ is connected and open subset of $A_m$ containing zero.
2. The generalization of the Taylor series of $f^{(m)}$ near zero (it must exist) is given by $f^{(m)}(x) = \sum_{n=0}^\infty P_n^{(m)}(x)$, where $P_n^{(m)}(x)$ is SMP.

Definition 2.4. A homogeneous SMP $P_n^{(m)}(x)$, $x \in A_m$ is a function of the form $P_n^{(m)}(x) = p_n^{(m)}(x) \alpha_n$, where $\alpha_n \in A_m$ and $p_n^{(m)}(x)$ is given by

$$p_n^{(m)}(x) = \frac{n!}{(m)_n} \sum_{k=0}^{n} \frac{(m)_k}{k!} \frac{x^k}{t^k}$$

(2.1)

where $(b) = b(b+1) \ldots (b+b-1)$ and $x \in A_m$, $b \in R$.

Remark 2.1. The SMP sequence $\{P_n^{(m)}(x)\}$ from an Appell sequence if it satisfies that $T_{\partial x_j}^{P_n^{(m)}(x)} = n \alpha_{n-1}(x)$.

For further information on monogenic functions, one can refer to [13,22,27–29].

Definition 2.5. A unitary left module $X$ over $A_m$ is an abelian group $(X, t)$ with a mapping $A_m \times X \to X$; $(\lambda, f^{(m)}) \mapsto \lambda f^{(m)}$ such that for all $\lambda, \mu \in X$ and $f^{(m)}, r^{(m)} \in A_m$ such that the following axioms hold:

1. $(\lambda + \mu) f^{(m)} = \lambda f^{(m)} + \mu f^{(m)}$,
2. $(\lambda \mu) f^{(m)} = \lambda (\mu f^{(m)})$,
3. $\lambda (f^{(m)} + r^{(m)}) = \lambda f^{(m)} + \lambda r^{(m)}$,
4. $e_0 f^{(m)} = f^{(m)}$.

A unitary right module can be defined analogously as follows.

Definition 2.6. A unitary right module $X$ over $A_m$ is an abelian group $(X, t)$ with a mapping $A_m \times X \to X$; $(f^{(m)}, \lambda) \mapsto \lambda f^{(m)}$ such that for all $\lambda, \mu \in X$ and $f^{(m)}, r^{(m)} \in A_m$ such that the following axioms hold:

1. $f^{(m)}(\lambda + \mu) = f^{(m)}\lambda + f^{(m)}\mu$,
2. $f^{(m)}(\lambda \mu) = (f^{(m)}\lambda)\mu$,
3. $(f^{(m)} + r^{(m)})\lambda = f^{(m)}\lambda + r^{(m)}\lambda$,
4. $f^{(m)} e_0 = f^{(m)}$.

It should be noted that if $X$ is a field, then a unitary right (resp. left) module is a linear space and if $R e_0 = A_0 \subset A_m$ then $X$ becomes a real linear space. In the sequel, all modules will be unitary right modules.

Let $X_R$ be a unitary right module over $A_m$. A function $\| \cdot \| : X_R \to [0, \infty)$, is called a seminorm on $X_R$, if it fulfills for all $f^{(m)}, r^{(m)} \in X_R$, $\lambda \in A_m$ and $k \in R$:

1. $\|f^{(m)} + r^{(m)}\| \leq \|f^{(m)}\| + \|r^{(m)}\|$. 

If \( P \) is a family of seminorms on \( X_{\mathbb{R}} \), then a subset \( Q \) of \( P \) is said to be a base of seminorms for \( P \), if for each \( p \in P \) there is a \( q \in Q \) and a positive real \( s \) such that \( p \leq sq \).

Definition 2.7. Let \( P \) be a family of countable proper system of seminorms on \( X_{\mathbb{R}} \).

1. If for any finite number \( p_{1}, p_{2}, \ldots, p_{l} \in P \), with \( l > 0 \), there are \( p \in P \) and \( C > 0 \) such that, for all \( f^{(m)} \in X_{\mathbb{R}} \), \( \sup_{k=1,\ldots,l} p_{k}(f^{(m)}) \leq Cp(f^{(m)}) \), then \( P \) is called a proper system of seminorms on \( X_{\mathbb{R}} \).

2. If \( (X_{\mathbb{R}}, P) \) is a seminormed Hausdorff topological space such that \( j < l \) implies that \( P_{j}(f^{(m)}) \leq P_{k}(f^{(m)}) \) for all \( f^{(m)} \in X_{\mathbb{R}} \) and for all \( f^{(m)} \in U \subseteq X_{\mathbb{R}} \), there are \( \epsilon > 0 \) and \( M > 0 \) such that \( \{g^{(m)} \in X_{\mathbb{R}} : P_{j}(f^{(m)} - r^{(m)}) \leq \epsilon \} \subseteq U \) for all \( j \leq M \) and \( X_{\mathbb{R}} \) is complete with respect to the metric topology, then \( X_{\mathbb{R}} \) is called Fréchet module.

3. A sequence \( \{P_{n}\}_{n \geq 0} \) in \( X_{\mathbb{R}} \) is said to converge to an element \( f^{(m)} \in X_{\mathbb{R}} \) if and only if, for all \( P_{j} \in P \), we have \( \lim_{n \to \infty} P_{j}(f^{(m)} - f^{(m)}) = 0 \).

Definition 2.8. A Banach module over \( X_{\mathbb{R}} \) is an \( X_{\mathbb{R}} \) over \( A_{m} \) equipped with a function \( ||| \cdot ||| : A_{m} \rightarrow [0, \infty) \) such that for all \( f^{(m)}, r^{(m)} \in A_{m} \), and \( \lambda \in X_{\mathbb{R}} \):

1. \( |||0|||_{A_{m}} = 0 \),
2. \( |||f^{(m)} + r^{(m)}|||_{A_{m}} \leq |||f^{(m)}|||_{A_{m}} + |||r^{(m)}|||_{A_{m}} \),
3. There exists \( c > 0 \) such that \( |||f^{(m)}\lambda|||_{A_{m}} \leq c |||\lambda|||_{A_{m}} \).
4. \( |||f^{(m)}|||_{A_{m}} = 0 \Rightarrow f^{(m)} = 0_{A_{m}} \),
5. \( A_{m} \) is complete with respect to the metric \( d(f^{(m)}, r^{(m)}) = |||f^{(m)} - r^{(m)}|||_{A_{m}} \).

It should be noted that a Banach module is a Fréchet module over \( A_{m} \).

Given an open ball \( S(R) \) and a closed ball \( \overline{S}(R) \) with both having a radius \( R > 0 \) and given an open ball \( S^{+}(R) \) enclosing the closed ball \( \overline{S}(R) \). We then consider: \( T[S(R)] = \{ r^{(m)} \in T[S(R)] : r^{(m)}(x) \text{ is SMF } \forall x \in S(R) \} \), \( T[\overline{S}(R)] = \{ r^{(m)} \in T[\overline{S}(R)] : r^{(m)}(x) \text{ is SMF } \forall x \in \overline{S}(R) \} \), \( T[S^{+}(R)] = \{ r^{(m)} \in T[S^{+}(R)] : r^{(m)}(x) \text{ is an entire SMF in } A_{m} \} \), \( T[0^{+}] = \{ r^{(m)} \in T[0^{+}] : r^{(m)}(x) \text{ is SMF at the origin} \} \). Then the countable family of a proper system of seminorms (which defines a Fréchet module) of each of the sets mentioned above are given respectively by: \( |||r^{(m)}|||_{R} = \sup_{S_{n}} |r^{(m)}(x)|, x \in A_{m}, \forall R' < R, r^{(m)} \in T[S(R)] \), \( |||r^{(m)}|||_{R} = \sup_{S_{n}} |r^{(m)}(x)|, x \in A_{m}, r^{(m)} \in T[\overline{S}(R)] \), \( |||r^{(m)}|||_{R} = \sup_{S_{n}(R')} |r^{(m)}(x)|, x \in A_{m}, \forall R < R', r^{(m)} \in T[S^{+}(R)] \), \( |||r^{(m)}|||_{R} = \sup_{S_{n}(R_{0})} |r^{(m)}(x)|, x \in A_{m}, \forall R' \in T[0^{+}] \).

Definition 2.9. Let \( \{P_{n}^{(m)}(x)\} \) be a sequence of Fréchet module \( X_{\mathbb{R}} \). Then \( P_{n}^{(m)}(x) \) is a base, if it can be expressed in the form

\[
P_{n}^{(m)}(x) = \sum_{k} P_{k}^{(m)}(\pi_{n,k}^{(m)}, \pi_{n,k}^{(m)}) \in A_{m}
\]

Where

\[
P_{n}^{(m)}(x) = \sum_{k} P_{k}^{(m)}(\pi_{n,k}^{(m)}, \pi_{n,k}^{(m)}) \in A_{m}
\]

\[
\Pi^{(m)} = (\pi_{n,k}^{(m)}) \text{ and } P_{n}^{(m)} = (P_{n,k}^{(m)}) \text{ are the Clifford matrices of operators and coefficients of the base } \{P_{n}^{(m)}(x)\} \text{ in } A_{m}.
\]

Remark 2.2. Every simple base is a base of degree \( n \in N \) and every Cannon base (i.e., \( \lim_{n \to \infty} \sup\{N_{n}\frac{1}{2} = 1 \), where \( N_{n} \) is the number of non-zero terms \( \pi_{n,k}^{(m)} \) in (2.2)) is also a base.

Theorem 2.1. [13] The necessary and sufficient condition for a SMP to be a base in \( S(R) \) is \( P^{(m)} \Pi^{(m)} = \Pi^{(m)} P^{(m)} = I, \) where 1 is unit matrix.

Remark 2.3. For Appell SMP, the Cauchy’s inequality in the neighborhood of \( S(R) \), can be written as

\[
|P_{n,i}^{(m)}| \leq \frac{|||P_{n}^{(m)}|||_{R}}{R^{i}}
\]

(2.4)

If \( r^{(m)}(x) = \sum_{n} P_{n}^{(m)}(x) a_{n}^{(m)}(x) \) is a SMF on a Fréchet module \( X_{\mathbb{R}} \), then one can write

\[
r^{(m)}(x) = \sum_{k} P_{k}^{(m)}(x) II_{n}(r^{(m)})
\]

(2.5)
where

\[ \Pi_n(r^m) = \sum_k \pi_n^{(m)}(r^m) \]  

(2.6)

\textbf{Definition 2.10.} A base \( \{P_n^{(m)}(x)\} \) in a Fréchet module \( X_\mathbb{R} \) is called effective, if the basic series (2.5) converges uniformly to \( r^m(x) \) on \( X_\mathbb{R} \).

The Cannon function \( \lambda_{P_{\infty}}(R) \) concerning the effectiveness of the base in the Fréchet module \( X_\mathbb{R} \) is given by

\[ \lambda_{P_{\infty}}(R) = \limsup_{n \to \infty} \left( \frac{\omega_{P_{\infty}}(R)}{n \log n} \right) \]  

(2.7)

Where

\[ \omega_{P_{\infty}}(R) = \sum_k \left| \pi_k^{(m)} \right|_R \]  

(2.8)

\[ \left| P_n^{(m)} \right|_R = \sup_{x \in [R]} |P_n^{(m)}(x)| \]  

(2.9)

\textbf{Theorem 2.2.} Let \( \{P_n^{(m)}(x)\} \) be a sequence of bases in Fréchet modules \( S(R) \), \( T \) \( S(R) \), \( T \) \( S(R^+) \), \( T[\infty] \) or \( T[0^+] \). Then \( \{P_n^{(m)}(x)\} \) is effective if and only if \( \lambda_{P_{\infty}}(R) = R \), \( \lambda_{P_{\infty}}(R') < R \forall R' < R \), \( \lambda_{P_{\infty}}(R^+) = R \), \( \lambda_{P_{\infty}}(R) < \infty \forall R < \infty \), or \( \lambda_{P_{\infty}}(0^+) = 0 \), respectively.

\textbf{Definition 2.11.} The order \( \rho_{P_{\infty}} \) and type \( \tau_{P_{\infty}} \) of a base \( \{P_n^{(m)}(x)\} \) of SMP are defined as follows:

\[ \rho_{P_{\infty}} = \lim\lim_{R \to \infty} \sup_{n \to \infty} \frac{\log \omega_{P_{\infty}}(R)}{n \log n} \]  

(2.10)

\[ \tau_{P_{\infty}} = \lim_{n \to \infty} e^{-\limsup \frac{\omega_{P_{\infty}}(R)}{n}} \]  

(2.11)

If a base \( \{P_n^{(m)}(x)\} \) is of order \( \rho_{P_{\infty}} \) (resp., type \( \tau_{P_{\infty}} \)), then it will represent in any \( S(R) \) every entire SMF of order \( \frac{1}{\rho_{P_{\infty}}} \) (resp., type \( \frac{1}{\tau_{P_{\infty}}} \)).

\textbf{Definition 2.12.} The \( T_\rho \)-property of SMP in \( T[S(R)] \), \( T[S(R)] \) or at the origin, where \( 0 < \rho < \infty \), means that the SMP represents all entire SMFs of order less than \( \rho \) in the same domain.

\textbf{Theorem 2.3.} A SMP has \( T_\rho \)-property in \( T[S(R)] \), \( T[S(R)] \) or at the origin iff \( \omega_{P_{\infty}}(R) \leq \frac{1}{\rho} \), \( \omega_{P_{\infty}}(R') \leq \frac{1}{\rho} \forall R' < R \) or \( \omega_{P_{\infty}}(0^+) \leq \frac{1}{\rho} \) respectively, where

\[ \omega_{P_{\infty}}(R) = \limsup_{n \to \infty} \frac{\log \omega_{P_{\infty}}(R)}{n \log n} \]  

(2.12)

For more properties, one can see [13,27].

3. \textbf{Exponential operators on SMPs}

We now, state the new classes of exponential derivative-type and integral-type operators in \( A_m \).

\textbf{Definition 3.1.} Let \( \{P_n^{(m)}(x)\} \) be a base. Then:

1. the exponential derivative operator \( \text{EXP}(0) \), is defined by

\[ \text{EXP}(0)P_n^{(m)}(x) = e^{x}P_n^{(m)}(x) \]  

(3.1)

2. (ii) the exponential integral operator \( \text{EXP}(\xi) \), is defined by

\[ \text{EXP}(\xi)P_n^{(m)}(x) = e^{x \xi}P_n^{(m)}(x) \]  

(3.2)

Now, for \( \theta^j = \theta^{j-1}, j \in \mathbb{N} \), we introduce a \( 0 \)-operator as \( \frac{\partial n}{\partial n} \), and for \( \xi^j = \xi^{j-1}, j \in \mathbb{N} \), we introduce a \( \xi \)-operator as \( \frac{\partial n}{\partial n} \).

Then from (2.3), we have

\[ \text{EXP}(\theta)P_n^{(m)}(x) = \sum_k e^{x \theta}P_k^{(m)}(x)P_n^{(m)} \]  

(3.3)

\[ \text{EXP}(\xi)P_n^{(m)}(x) = \sum_k e^{x \xi}P_k^{(m)}(x)P_n^{(m)} \]  

(3.4)

In the sequel, we shall write \( \{E_n^{(m,0)}(x)\} \) (resp., \( \{E_n^{(m,\xi)}(x)\} \) abbreivation of \( \{\text{EXP}(0)P_n^{(m)}(x)\} \) (resp., \( \{\text{EXP}(\xi)P_n^{(m)}(x)\} \)).

Now, we establish the following theorem:

\textbf{Theorem 3.1.} Let \( \{P_n^{(m)}(x)\} \) be a base of SMPs in \( A_m \).

Then \( \{E_n^{(m,0)}(x)\} \) is also a base.

\textbf{Proof.} Since every matrix of coefficients (resp., operators) of \( \{E_n^{(m,0)}(x)\} \) can be expressed as

\[ E_n^{(m,0)} = \left( \sum_{k} \nabla_n^{(m)} \right) = \left( \sum_{k} \nabla_{n,k}^{(m)} \right) = \left( \sum_{k} \nabla_{n,k}^{(m)} \right) = \left( \sum_{k} \nabla_{n,k}^{(m)} \right) = I. \]
Theorem 4.4. Both \( E_n^{(m, \theta)}(\chi) \) and \( E_n^{(m, \xi)}(\chi) \) are effective for \( T \left[ S(R) \right] \), \( T \left[ S(R^+) \right] \), \( T \left[ 0^+ \right] \) and \( T \left[ \infty \right] \) spaces, if the following conditions are satisfied:

1. the base \( \{P_n^{(m)}(\chi)\} \) of SMP is effective for \( T \left[ S(R) \right] \), \( T \left[ S(R^+) \right] \), \( T \left[ 0^+ \right] \) and \( T \left[ \infty \right] \), respectively,
2. \( \lim_{n \to \infty} D_n = 1 \), where \( D_n \) is the highest degree of \( (2.2) \).

The proof of Theorem 4.2 is similar to [30] and Theorem 4.1, so we omit the details.

We give an example to show that the hypothesis (ii) in Theorem 4.1 is necessary.

Example 4.1. Suppose \( \{P_n^{(m)}(\chi)\} \) is a base of SMP such that

\[
P_n^{(m)}(\chi) = \begin{cases} p_n^{(m)}(\chi), & n \in \mathbb{E}, \\ p_n^{(m)}(\chi) + p_1^{(m)}(\chi), & l = 2n, n \in \mathbb{O}. \end{cases}
\]

If \( n \in \mathbb{E} \), then

\[
p_n^{(m)}(\chi) = p_n^{(m)}(\chi) \Rightarrow \omega_{P_n^{(m)}}(R) = R^n \Rightarrow \omega_{P_n^{(m)}}(1) = 1,
\]

when \( R=1 \Rightarrow \lim_{n \to \infty} \left\{ \omega_{P_n^{(m)}}(1) \right\}^{\frac{1}{R^n}} = 1 \)

If \( n \in \mathbb{O} \), then

\[
p_n^{(m)}(\chi) = P_n^{(m)}(\chi) - P_1^{(m)}(\chi) \Rightarrow \omega_{P_n^{(m)}}(R) = R^n + 2R^l \Rightarrow \omega_{P_n^{(m)}}(1) = 3,
\]

when \( R=1 \Rightarrow \lim_{n \to \infty} \left\{ \omega_{P_n^{(m)}}(1) \right\}^{\frac{1}{R^n}} = 1 \).

Hence \( \{P_n^{(m)}(\chi)\} \) is effective for \( T \left[ S(1) \right] \). Construct the set \( \{E_n^{(m, \theta)}(\chi)\} \) as follows:

\[
E_n^{(m, \theta)}(\chi) = \begin{cases} e_n p_n^{(m)}(\chi), & n \in \mathbb{E}, \\ e_n p_n^{(m)}(\chi) + e_1 p_1^{(m)}(\chi), & l = 2n, n \in \mathbb{O}. \end{cases}
\]

If \( n \in \mathbb{E} \), then

\[
p_n^{(m)}(\chi) = \frac{1}{e_n} E_n^{(m, \theta)}(\chi) \Rightarrow \omega_{E_n^{(m, \theta)}}(R) = R^n \Rightarrow \omega_{E_n^{(m, \theta)}}(1) = 1,
\]

when \( R=1 \Rightarrow \lim_{n \to \infty} \left\{ \omega_{E_n^{(m, \theta)}}(1) \right\}^{\frac{1}{R^n}} = 1 \)

If \( n \in \mathbb{O} \), then

\[
p_n^{(m)}(\chi) = \frac{1}{e_n} \left( E_n^{(m, \theta)}(\chi) - E_{n-1}^{(m, \theta)}(\chi) \right) \Rightarrow \omega_{E_n^{(m, \theta)}}(R) = R^n + 2e R^l \Rightarrow \omega_{E_n^{(m, \theta)}}(1) = 1 + 2e.
\]

Hence \( \{E_n^{(m, \theta)}(\chi)\} \) is not effective for \( T \left[ S(1) \right] \).
As an application of our results, we consider the following bases \( \{P_n^{(m)}(x)\} \) and \( \{Q_n^{(m)}(x)\} \) of the Bessel and generalized Bessel polynomials in \( T[\mathbb{S}(R)] \), respectively:

\[
P_0^{(m)}(x) = 1, \quad P_n^{(m)}(x) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)2^k} x^k; \quad (n \geq 1),
\]

\[
Q_0^{(m)}(x) = 1, \quad Q_n^{(m)}(x) = 1 + \sum_{k=0}^{n} \frac{n!(n+b-1)(n+b)\ldots(n+k-1)(n+k-b)}{n!(n-k)!} x^k \times P_k^{(m)}(x).
\]

According to [6,7], we have that both \( \{P_n^{(m)}(x)\} \) and \( \{Q_n^{(m)}(x)\} \) are effective for \( T[\mathbb{S}(R)] \) and condition (ii) of Theorem 4.1 holds. Then as an immediate application of Theorem 4.1 and Remark 4.1, we conclude that \( \{P_n^{(m)}(x)\} \), \( \{P_n^{(m,\ell)}(x)\} \), \( \{Q_n^{(m)}(x)\} \), and \( \{Q_n^{(m,\ell)}(x)\} \) are also effective for \( T[\mathbb{S}(R)] \).

5. Computations of order, type and \( T_\rho \)-property

In this section, we study the relationship between \( \rho_{P(n)} \), \( \tau_{P(n)} \) of \( \{P_n^{(m)}(x)\} \) and \( \rho_{E(n,\rho)} \), \( \tau_{E(n,\rho)} \) of \( \{E_n^{(m,\rho)}(x)\} \).

**Theorem 5.1.** If \( \{P_n^{(m)}(x)\} \) is a SMPs of order \( \rho_{P(n)} \) and type \( \tau_{P(n)} \) such that

\[
D_n = O[n] \quad (5.1)
\]

Then the following statement holds:

\[
\rho_{E(n,\rho)} \leq \rho_{P(n)} \quad \text{and} \quad \tau_{E(n,\rho)} \leq \tau_{P(n)} \quad \text{whenever} \quad \rho_{P(n)} = \rho_{P(n)},
\]

where \( \rho_{E(n,\rho)} \) is the order (resp., type) of \( \{E_n^{(m,\rho)}(x)\} \).

**Proof.** It follows from (2.10) and (4.2), that:

\[
\lim \limsup_{R \to \infty} \frac{\log \omega_{E(n,\rho)}(R)}{n \log n} \quad \text{and} \quad \lim \limsup_{R \to \infty} \frac{\log 2^m e^{D_n - n}(D_n + 1) + \log \omega_{E(n,\rho)}(R)}{n \log n}.
\]

This implies that \( \rho_{E(n,\rho)} \leq \rho_{P(n)} \).

From \( \rho_{E(n,\rho)} = \rho_{P(n)} \) and (2.11), we have

\[
\lim e^{\limsup_{R \to \infty} \frac{\omega_{E(n,\rho)}(R)^{1/\tau_{P(n)}}}{\rho_{E(n,\rho)}}} \leq \lim e^{\limsup_{R \to \infty} \frac{\omega_{E(n,\rho)}(R)^{1/\tau_{P(n)}}}{\rho_{P(n)}}}.
\]

which gives \( \tau_{E(n,\rho)} \leq \tau_{P(n)} \). Thus the theorem is proved.

Giving an example which supports Theorem 5.1:

**Example 5.1.** Suppose \( \{P_n^{(m)}(x)\} \) is a base of SMP such that \( P_0^{(m)} = 1, P_n^{(m)} = n^m + p_n^{(m)}(x) \).

Then

\[
\omega_{E(n,\rho)}(R) = 2n^m + R_n^m,
\]

\[
\rho_{P(n)} = \lim \limsup_{R \to \infty} \frac{\log(2n^m + R_n^m)}{n \log n} = 1,
\]

\[
\tau_{P(n)} = \lim e^{\limsup_{R \to \infty} \frac{2n^m + R_n^m}{n \log n}} = e.
\]

Constructing the corresponding \( \{E_n^{(m,\rho)}(x)\} \) base as following: \( E_0^{(m,\rho)} = 1, E_n^{(m,\rho)} = n^m + e^{p_n^{(m,\rho)}(x)} \).

Then

\[
\omega_{E(n,\rho)}(R) = \frac{2n^m}{e^{p_n^{(m,\rho)}(x)}} + R_n^m,
\]

So that

\[
\rho_{E(n,\rho)} = \lim \limsup_{R \to \infty} \frac{\log \left( \frac{2n^m}{e^{p_n^{(m,\rho)}(x)}} + R_n^m \right)}{n \log n} = 1,
\]

\[
\tau_{E(n,\rho)} = \lim e^{\limsup_{R \to \infty} \frac{2n^m + R_n^m}{n \log n}} = e.
\]

Hence the two bases \( \{P_n^{(m)}(x)\} \) and \( \{E_n^{(m,\rho)}(x)\} \) have the same order 1 and type \( e \) in \( T[\mathbb{S}(R)] \).

Next example shows the importance of condition (5.1).

**Example 5.2.** Suppose \( \{P_n^{(m)}(x)\} \) is a base of SMP in such that

\[
P_n^{(m)}(x) = \begin{cases} P_n^{(m)}(x), & x \in \mathbb{E}, \\ P_n^{(m)}(x) + \frac{v}{b^2} P_2^{(m)}(x), & v = n^m, R = b \in \mathbb{E}. \end{cases}
\]

If \( n \in \mathbb{O} \), then

\[
P_n^{(m)}(x) = P_n^{(m)}(x) - \frac{v}{b^2} P_2^{(m)}(x) = \omega_{E(n,\rho)}(R)
\]

\[
= R^m + 2v \left( \frac{R}{b} \right)^{2v}, \quad \text{but} \quad R = b = \omega_{E(n,\rho)}(R) = R^m + 2v
\]

\[
\Rightarrow \rho_{P(n)} = \lim \limsup_{R \to \infty} \frac{\log(R^m + 2v)}{n \log n} = 1.
\]

Hence \( \{P_n^{(m)}(x)\} \) is of order 1 in \( T[\mathbb{S}(R)] \), construct the set \( \{E_n^{(m,\rho)}(x)\} \) as follows:
Remark 5.1. The following theorem establishes the $T_r$-$\rho$-property of order $1$.

Theorem 5.2. Let $\omega_{E^{(m,\theta)}}(R) = \limsup_{n \to \infty} \frac{\log \omega_{E^{(m,\theta)}}(n)}{n \log n}$.

Using (4.2), (12.12) and Theorem 5.2, we have

$$\omega_{E^{(m,\theta)}}(R) \leq \omega_{P^{(m,\theta)}}(R) \leq \frac{1}{\rho}.$$  

5.1. Conclusions

In this paper, we have established a novel set of polynomial bases in $A_m$ through the utilization of exponential derived and integral operators in Clifford analysis. The operators can be viewed as a generalization of the complex form $C$ (when $m = 1$) and quaternion form $H$ (when $m = 2$). We investigated the convergence properties of the effectiveness of operators $E^{(m,\theta)}(x)$ and $E^{(m,\ell)}(x)$, analyzing their order, type and $T_r$-$\rho$-property. Additionally, we provided illustrative examples and applications to elucidate the principal findings.
References