[Al-Azhar Bulletin of Science](https://absb.researchcommons.org/journal)

Volume 34 | Issue 1 Article 7

2023

Section: Astronomy and Meteorology

A NUMERICAL APPROACH TO SOLVE THE INITIAL-VALUE PROBLEM OF TWO-BODY WITH UNIVERSAL VARIABLE

Ahmed B. Yassen
Space Science Dep., Fac., of navigation Science and Space Technology (NSST), Beni-Suef Uni., Egypt, ahmed_hafez@azhar.edu.eg

Waleed N. Ahmed
Applied Mathematics Dep., National Research Centre (NRC), Egypt

Ahmed H. Ibrahim
Astronomy and Meteorology Dep., Fac. Sci., Al-Azhar Uni., Egypt

Abdelaziz A. Bakry
Astronomy and Meteorology Dep., Fac. Sci., Al-Azhar Uni., Egypt

Hany R. Dwidar
Astronomy, Space and meteorology Dep., Fac. Sci., Cairo Uni., Egypt

How to Cite This Article

Yassen, Ahmed B.; Ahmed, Waleed N.; Ibrahim, Ahmed H.; Bakry, Abdelaziz A.; and Dwidar, Hany R. (2023) "A NUMERICAL APPROACH TO SOLVE THE INITIAL-VALUE PROBLEM OF TWO-BODY WITH UNIVERSAL VARIABLE," *Al-Azhar Bulletin of Science*: Vol. 34: Iss. 1, Article 7.
DOI: https://doi.org/10.58675/2636-3305.1640

ORIGINAL ARTICLE

A Numerical Approach to Solve the Two-body Initial-value Problem with a Universal Variable

Ahmed Badwy Yassen [a], Waleed Hafez Ahmed [b], Ahmed Hafez Ibrahim [c,*], Abdelaziz Ahmed Bakry [c], Hany Ramadan Dwidar [d]

[a] Space Science Department, Faculty of Navigation Science and Space Technology (NSST), Beni-Suef University, Egypt
[b] Applied Mathematics Department, National Research Centre (NRC), Egypt
[c] Astronomy and Meteorology Department, Faculty of Science, Al-Azhar University, Egypt
[d] Astronomy, Space and Meteorology Department, Faculty of Science, Cairo University, Egypt

Abstract

The solution of the two-body initial-value problem of gives inaccurate final state predictions for the orbital motions of artificial satellites. This is due to the presence of singularities and the poor selection of variables. In the current study, we numerically investigated the initial-value problem using the universal anomaly approach. To clarify the problem under concern, we carried out several numerical examples using a homemade software package. We considered five space missions, around the two planets Earth and Venus, which represent circular, near-circular, and elliptic orbits. We showed that the universal anomaly approach facilitates the numerical and analytical treatments of the two-body dynamics and works equally well for different types of orbits. Moreover, we developed a computation algorithm to handle the perturbed problem in cylindrical coordinates for the initial value problem taking into consideration the geopotential of the two planets up to the third zonal harmonic J_3 and the tesseral coefficient C_{22}.

Keywords: Cylindrical coordinate, Initial value problem, Perturbation, Trajectories, Venus

1. Introduction

In classical mechanics, the two-body initial-value problem may be defined as: At an instant of time, we give certain suitable initial conditions for the involved quantities, such as the velocity and position, to predict the subsequent motion. The two-body problem describes the dynamics of two celestial objects in close proximity, abstractly viewed as point masses. The problem assumes that the two bodies interact only through their mutual gravitational potential, and all other forces are ignored. The problem deals with the orbital and rotational motion of two finite bodies [1].

Beyond just the field of astrodynamics, the two-body problem has broad applicability in numerous engineering and scientific fields. One example of such a system is an artificial satellite that rotates around the Earth. Under the mentioned assumptions, the problem is simple and represents the only integrable system in celestial mechanics. In the case of nonspherical mass distributions of either one of the bodies, the problem becomes non-integrable and can exhibit chaotic dynamics. Since the time of Newton and Kepler, the solution of the classical gravitational two-body problem has been completely obtained. However, due to the existence of a singularity and deficiency of choice of variables, inconveniences and even difficulties occur when using classical methods. The formulation of the equations of motion is different depending on whether the conic section is a hyperbola, a parabola, or an ellipse. Definite formulae are required to determine the position for any case. When dealing with the same orbit for a long time they give satisfactory results. However, the orbit may suffer qualitative and quantitative changes in the computations of interplanetary trajectories. On the other hand, the conventional equations of motion of space

Received 17 August 2022; revised 10 October 2022; accepted 31 October 2022.
Available online 3 October 2023

* Corresponding author at: Astronomy and Meteorology Department, Faculty of Science, Al-Azhar University, Cairo, 11787, Egypt.
E-mail address: ahmed_hafez@azhar.edu.eg (A.H. Ibrahim).

https://doi.org/10.21608/2636-3305.1640

Table 1. Earth's satellite cartesian coordinate system.

	x (km)	y (km)	z (km)	v_x (km.s^{-1})	v_y (km.s^{-1})	v_z (km.s^{-1})
Ablestar 008	3006.76	−6550.8	12.5658	2.66687	1.25074	6.8602
Altair	7582.87	−2218.92	8.0032	1.30561	4.6648	5.2367
Vanguard3	−7792.91	1302.12	4.58985	0.166568	−6.09413	3.93551

Table 2. Earth's satellites classical orbital elements.

	(e)	(a) (km)	(i) (deg)	(ω) (deg)	(Ω) (deg)	(M) (deg)
Ablestar 008	0.0080970	7265.5	66.7634°	11.9263°	294.6118°	348.3713°
Altair	0.0113060	7970	47.2328°	40.3825°	343.6356°	320.5292°
Vanguard3	0.1664214	8262	33.3392°	84.2892°	170.4635°	294.4242°

Table 3. Venus's satellites cartesian coordinate system.

	x (km)	y (km)	z (km)	v_x (km.s^{-1})	v_y (km.s^{-1})	v_z (km.s^{-1})
Venera8	5603.52	2405.54	1767.59	−3.23399	3.72609	5.35745
Venera15	6494.05	1039.07	2391.12	−2.35075	−0.80682	8.84907

Table 4. Venus's satellites orbital elements.

	(e)	(a) *Km*	(i) *deg*
Venera8	0.03732	6591	51.7°
Venera15	0.8211	38,848	92.5°

dynamic are unstable in the Lyapunov sense. Thus, for the motion of spacecraft's, the solution of these equations gives inaccurate predictions. To avoid these inconveniences, several authors introduced successful special 'universal' variables that regularize the equations of motion of dynamics and maintain the same form of the solution in all cases [2—5]. Due to the importance of the current problem, it has been studied for decades. During last decades, studies were performed by several authors, they used numerical or approximation methods for solution of the Kepler's equation. The earliest paper by Stumpff [6] and Herrick introduced a different formulation through the manipulation of series expansion of conventional conic variables [7]. Sperling obtained a related set of universal variables from the transformed equations of motion [8].

Mortari and Elipe [9] proposed a new approach to solve Kepler's equation based on the use of implicit functions. The proposed method, with limited computational capability, is particularly suitable for space-based applications. Tokis [10] obtained a solution of the universal Kepler's equation in closed form with the help of the two-dimensional Laplace technique, expressing the universal functions as a function of the universal anomaly and time. Wisdom and Hernandez [11] derived and present a fast and accurate solution of the initial value problem for Keplerian motion in universal variables that does

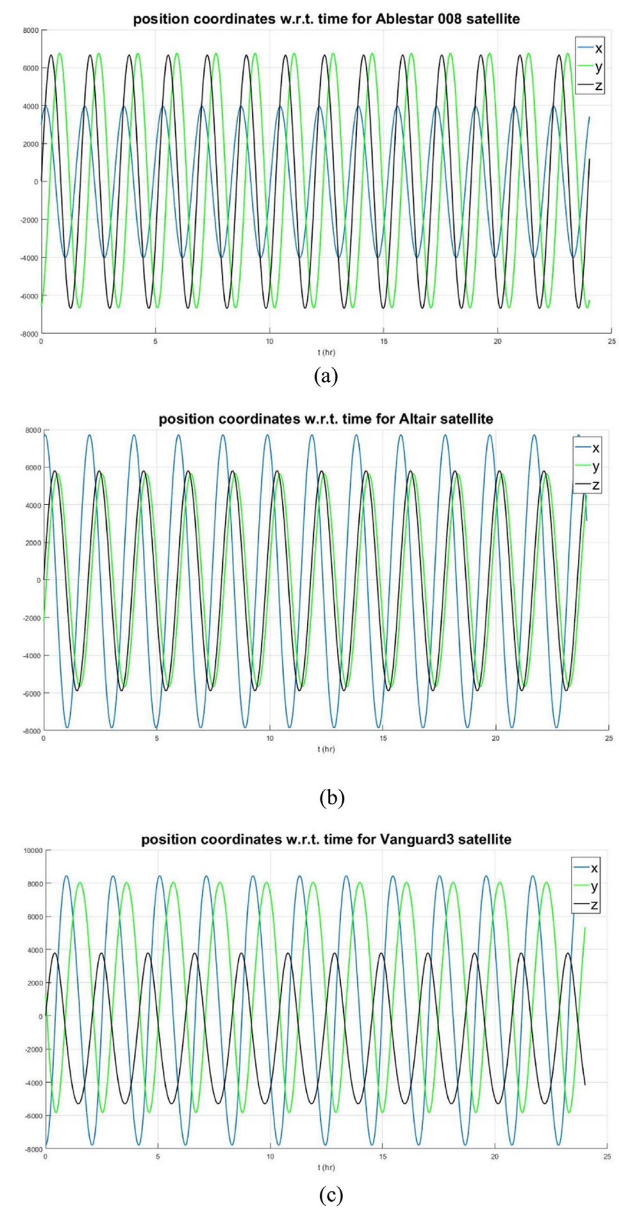
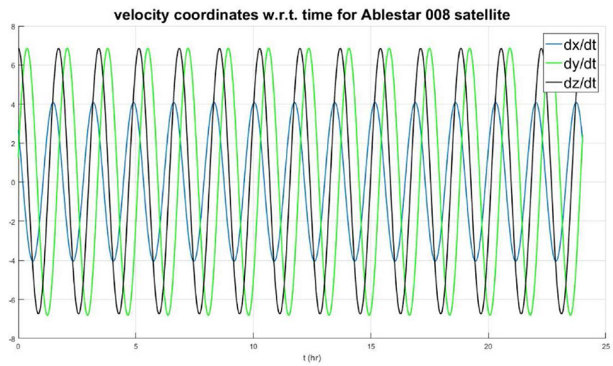
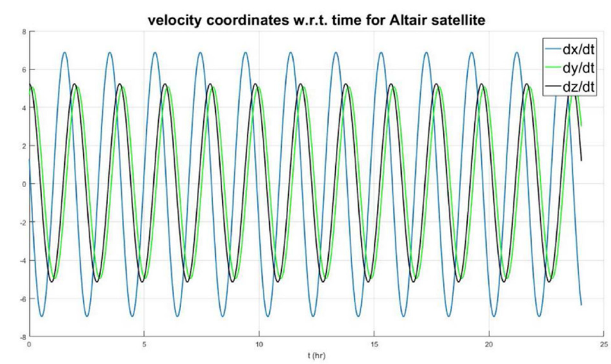


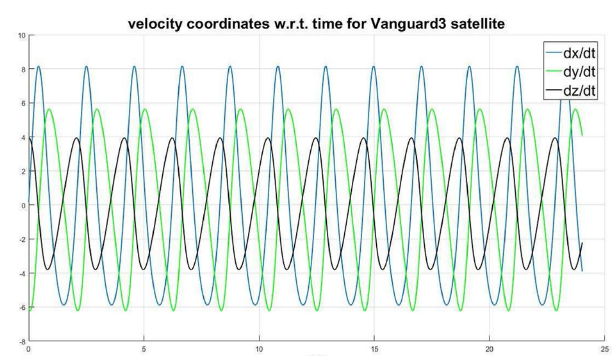
Fig. 1. The position of the different Earth's satellites $x(t)$, $y(t)$ and $z(t)$ as a function of time.



(a)

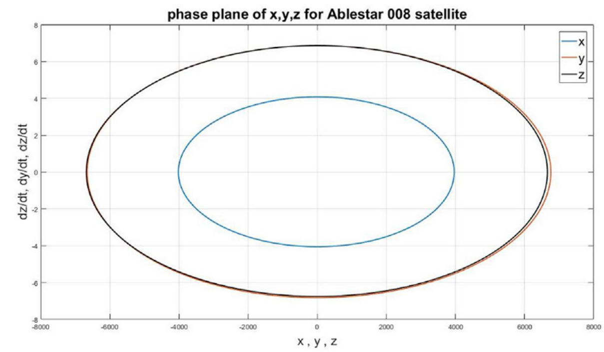


(b)

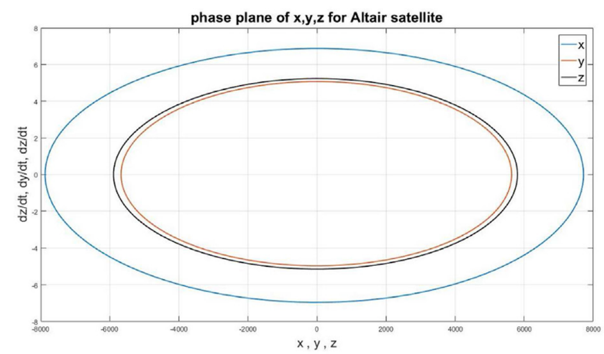


(c)

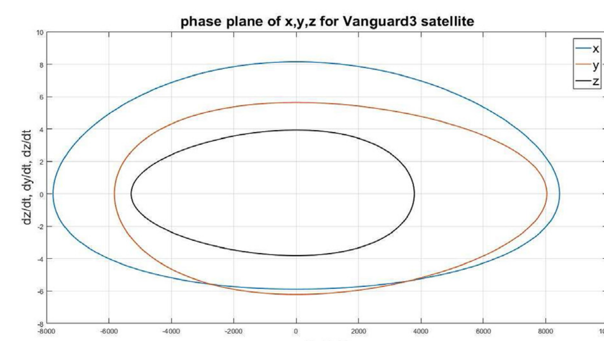
Fig. 2. The velocity of the different Earth's satellites $\dot{x}(t), \dot{y}(t)$ and $\dot{z}(t)$ as a function of time.



(a)



(b)



(c)

Fig. 3. The phase space plane of different Earth's satellites.

not use the Stumpff series. They found that it performs better than methods based on the Stumpff series. Recently, Pulido and Peláez [12] introduced a new approach for solving the Kepler equation for hyperbolic orbits. The authors tried to substantially improve well-known classic schemes based on the excellent properties of the Newton−Raphson iterative methods. The developed code provides fast and accurate solutions. Sharaf and Dwidar [2] set up the initial value problem of space dynamics in universal Stumpff anomaly and developed an analytical and computational approach. In the present study, we addressed numerically the two-body initial-value problem using the universal anomaly approach. We performed several numerical simulations to clarify the problem.

2. Universal formulation

The classical equations of the two-body problem cause troubles when a transition from one kind of orbit to another occurs. Therefore, to make the initial-value problem free from such troubles the relevant two-body equations are derived utilizing universal functions. To avoid using multiple formulations to describe motion in different orbits, the problem is generalized using transcendental

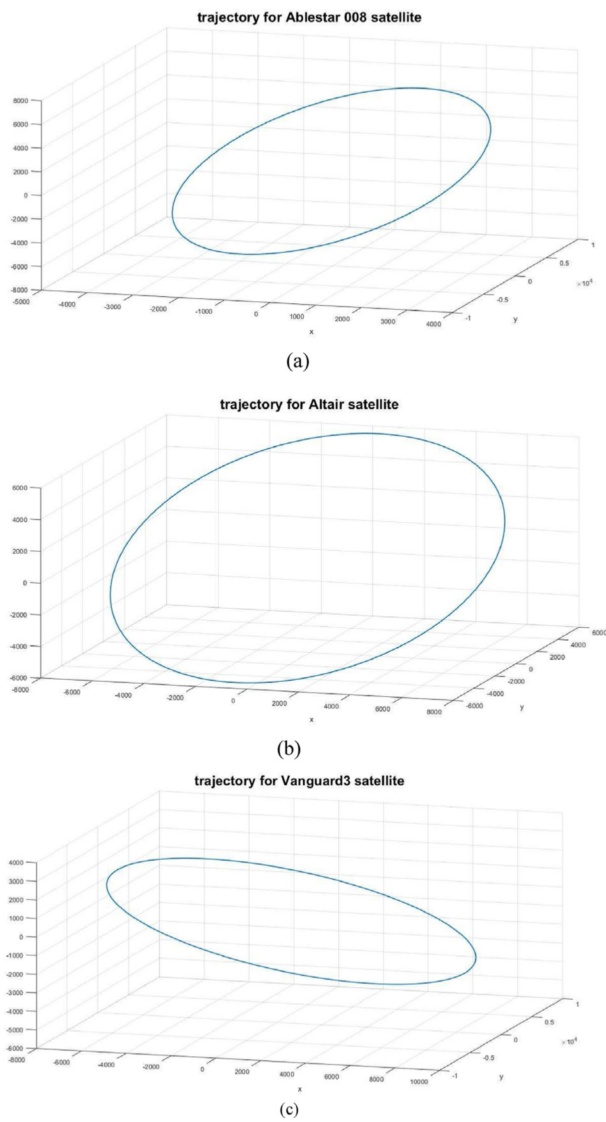


Fig. 4. The trajectory in space of the different Earth's satellites.

functions. Utilizing these functions, applicable general formulae can be obtained that are valid at the same time for all types of conic sections.

Now the key differential relationships can be summarized as follows [13]:

$$df = \frac{1}{r}\begin{cases} p\, d\left(\tan\frac{f}{2}\right) \\ b\, dE \\ b\, dH \end{cases} = \frac{h}{r^2}dt \tag{1}$$

Because, for the three types of conic sections, we have

$$\frac{h}{p} = \sqrt{\frac{\mu}{p}}, \frac{h}{b} = \sqrt{\frac{\mu}{a}}, \frac{h}{b} = \sqrt{\frac{\mu}{-a}}$$

then we can write

$$\sqrt{\mu}dt = r\begin{cases} d\left(\sqrt{p}\,\tan\frac{f}{2}\right) \\ d\,(\sqrt{a}\,E) \\ d\,(\sqrt{-a}\,H) \end{cases} = rdx \tag{2}$$

where f is the true anomaly, E the eccentric anomaly, μ the gravitational parameter, H the hyperbolic anomaly, a the semi-major axis, p the semi-latus rectum, h the angular momentum, **r** the radius vector, and b is the semiminor axis. The variable χ can be considered as a new and independent variable, (i-e) a kind of generalized anomaly. Therefore, the nonlinear equations of motion can be transformed into linear differential equations with constant coefficients, when χ is used as an independent variable instead of time. The transformation defined by

$$\sqrt{\mu}\,\frac{dt}{dx} = r \tag{3}$$

is called the Sundman transformation. Now the values $\mathrm{r}, r, \sigma = \frac{dr}{d\chi}$ and t can all be found as solutions of simple ordinary differential equations.

To do that, let us differentiate the identity

$$r^2 = \mathbf{r}.\mathbf{r}$$

and obtain

$$r\frac{dr}{d\chi} = \frac{dt}{d\chi}\mathbf{r}.\frac{d\mathbf{r}}{dt} = r\sigma$$

From Lagrangian coefficients

$$\sigma = \frac{\mathbf{r}.\mathbf{v}}{\sqrt{\mu}} = \sqrt{p}\tan\frac{1}{2}f$$

$$r\frac{dr}{d\chi} = r\sigma$$

by cancelling the factor r and differentiating the above equation, we have

$$\frac{d^2r}{d\chi^2} = \frac{d\sigma}{d\chi} = \frac{r}{\mu}\frac{d}{dt}(\mathbf{r}.\mathbf{v}) = \frac{r}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} - \frac{\mu}{r}\right) = 1 - \frac{r}{a}$$

It is convenient here and subsequently to write α for the reciprocal of a, we have

$$\alpha \equiv \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

and may be positive, negative, or zero. In summary, then

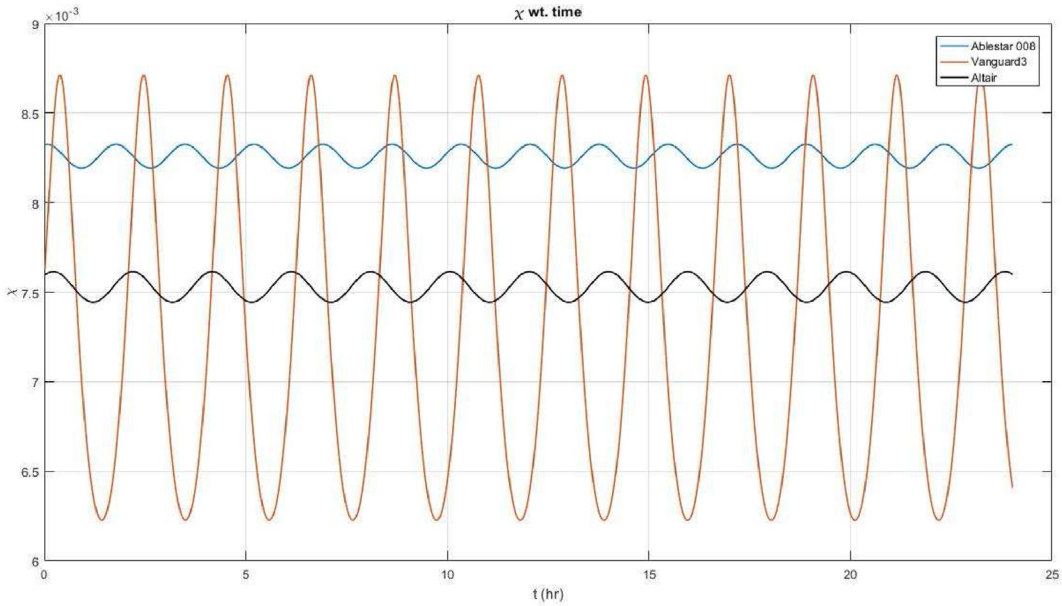


Fig. 5. The universal anomaly of different Earth's satellites as a function of time.

$$\frac{dr}{d\chi} = \sigma = \sqrt{\mu}\,\frac{d^2t}{d\chi^2}$$

$$\frac{d^2r}{d\chi^2} = \frac{d\sigma}{d\chi} = 1 - \alpha r$$

$$\frac{d^3r}{d\chi^3} = \frac{d^2\sigma}{d\chi^2} = \sqrt{\mu}\,\frac{d^4t}{d\chi^4} = -\alpha\sqrt{\mu}\,\frac{d^2t}{d\chi^2}$$

so that σ, r, and t are solutions of the equations

$$\frac{d^2\sigma}{d\chi^2} + \alpha\sigma = 0, \frac{d^3r}{d\chi^3} + \alpha\frac{dr}{d\chi} = 0, \frac{d^4t}{d\chi^4} + \alpha\frac{d^2t}{d\chi^2} = 0 \quad (4)$$

The derivatives of the position vector $\mathbf{r}$ are given by

$$\frac{d\mathbf{r}}{d\chi} = \frac{r}{\sqrt{\mu}}\mathbf{v}, \frac{d^2\mathbf{r}}{d\chi^2} = \frac{\sigma}{\sqrt{\mu}}\mathbf{v} - \frac{1}{r}\mathbf{r}$$

lead in a similar manner to

$$\frac{d^3\mathbf{r}}{d\chi^3} + \alpha\frac{d\mathbf{r}}{d\chi} = 0 \quad (5)$$

Equations (4) and (5) represent a set of linear differential equations with constant coefficients, for which solutions can be found without difficulty. Nevertheless, it is useful in this case to construct the solutions in a form using a family of special universal functions.

2.1. The universal functions $U_n(\chi;\alpha)$

These functions are first introduced by [2] as

$$U_n(\chi;\alpha) = \chi^n \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha\chi^2)^k}{(n+2k)!}$$

Then the general form of Kepler equation was obtained as

$$\sqrt{\mu}(t - t_0) = r_0 U_1(\chi;\alpha) + \sigma_0 U_2(\chi;\alpha) + U_3(\chi;\alpha)$$

together with

$$r = r_0 U_0(\chi;\alpha) + \sigma_0 U_1(\chi;\alpha) + U_2(\chi;\alpha)$$
$$\sigma = \sigma_0 U_0(\chi;\alpha) + (1 - \alpha r_0) U_1(\chi;\alpha)$$

The expressions for the Lagrangian coefficients are given as

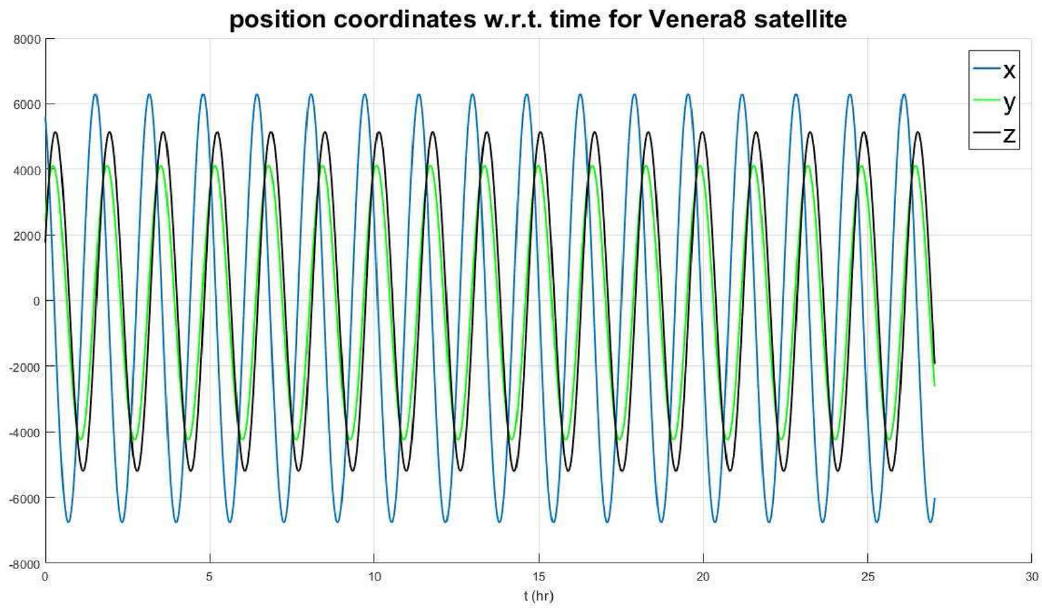
$$F = 1 - \frac{1}{r_0}U_2(\chi;\alpha) \quad G = \frac{r_0}{\sqrt{\mu}}U_1(\chi;\alpha) + \frac{\sigma_0}{\sqrt{\mu}}U_2(\chi;\alpha)$$
$$F_t = -\frac{\sqrt{\mu}}{r r_0}U_1(\chi;\alpha) \quad G_t = 1 - \frac{1}{r}U_2(\chi;\alpha) \quad (6)$$

We get the position and velocity of the satellite

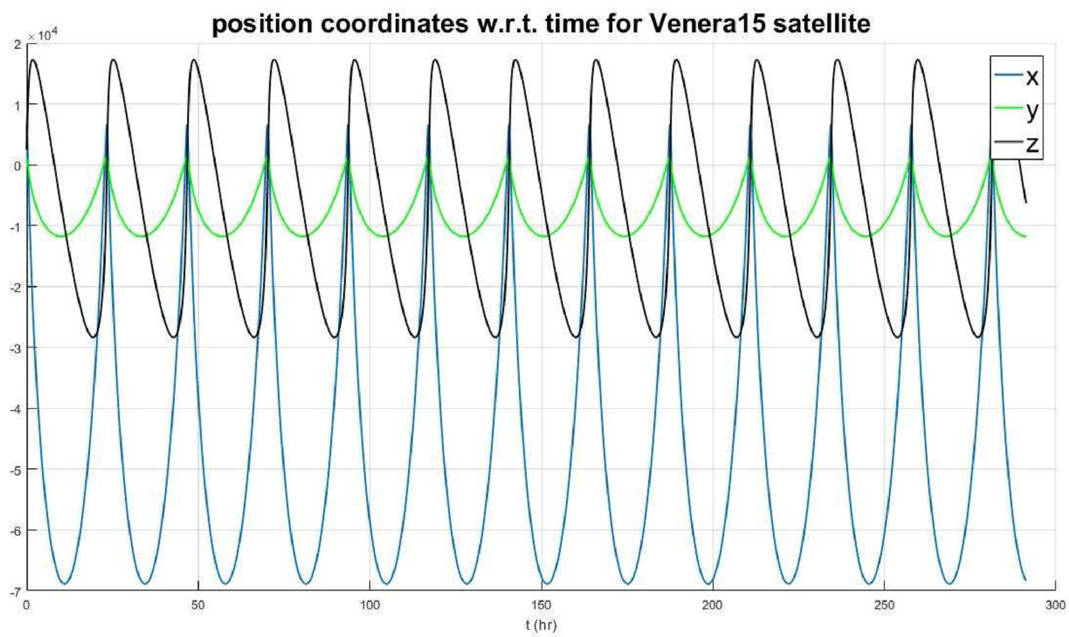
$$\mathbf{r} = F\mathbf{r}_0 + G\mathbf{v}_0$$

$$\mathbf{v} = F_t\mathbf{r}_0 + G_t\mathbf{v}_0$$

The above equation (6) are termed 'universal' since it is void of singularities and valid for any conic sections.



(a)



(b)

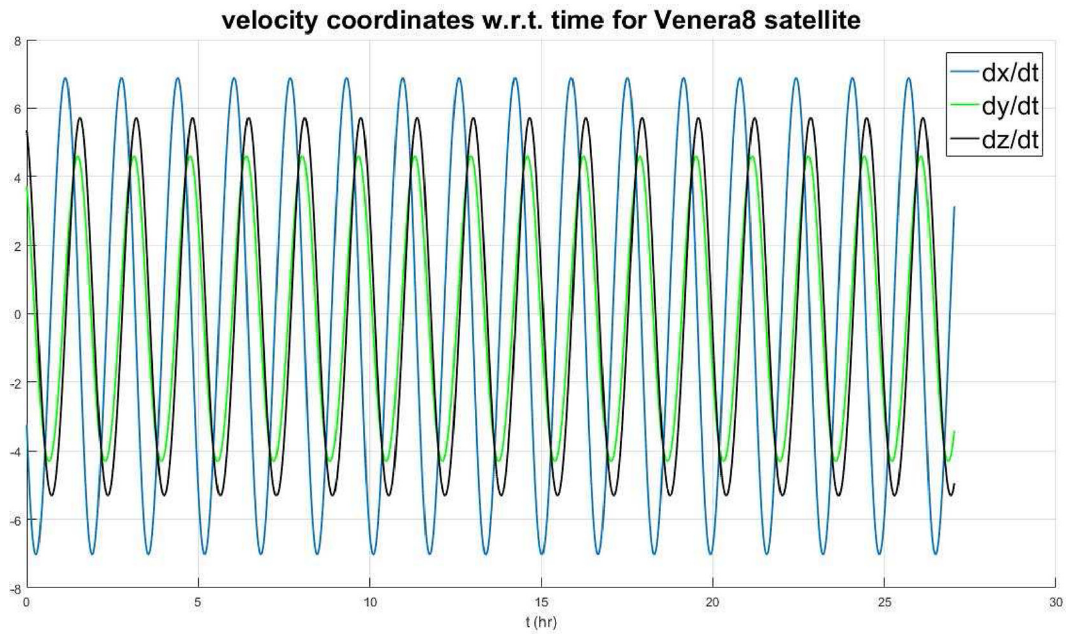
Fig. 6. The position of the different Venus's satellites $x(t), y(t)$ and $z(t)$ as a function of time.

3. Analytical formulation of cylindrical coordinates

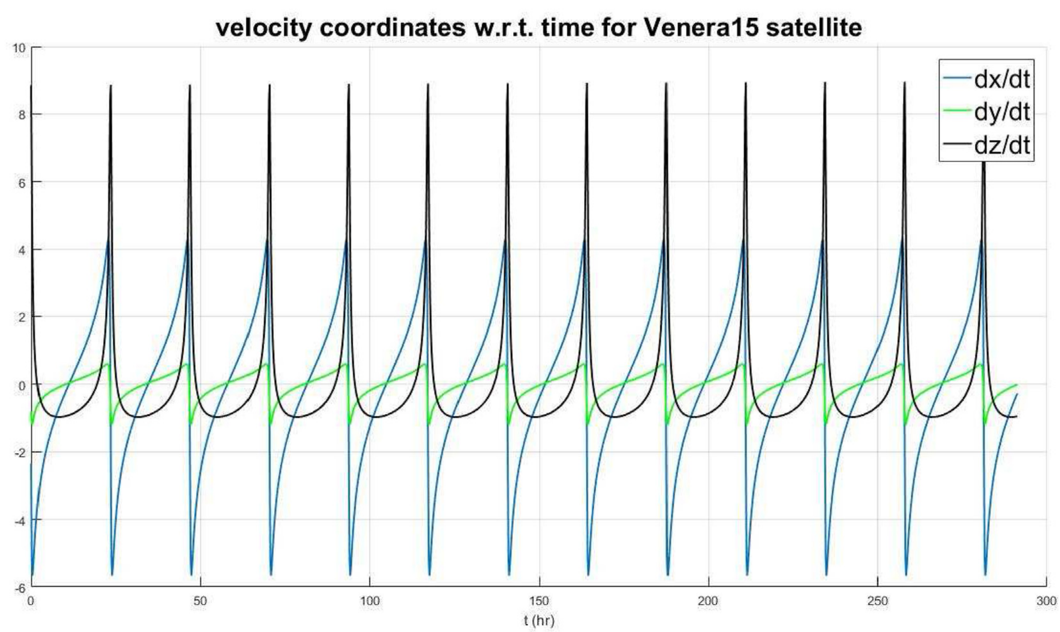
In this section, the two-body initial-value problem of will be formulated utilizing cylindrical coordinates. We will develop a computation algorithm for the initial value problem of J_2, J_3, and C_{22} gravity perturbed trajectories. Some numerical applications are carried out, for selected test orbits, for the problem of final state prediction.

3.1. Coordinate, velocity transformations

Let the rectangular coordinates (x, y, z) of any point be expressed as a function of the cylindrical coordinate (ρ, θ, Z) so that



(a)



(b)

Fig. 7. The Velocity of the different Venus's satellites $\dot{x}(t), \dot{y}(t)$ and $\dot{z}(t)$ as a function of time.

$x = \rho\cos\theta, y = \rho\sin\theta, z = Z$

where

$0 \le \rho < \infty, -\pi < \theta \le \pi, -\infty < Z < \infty$

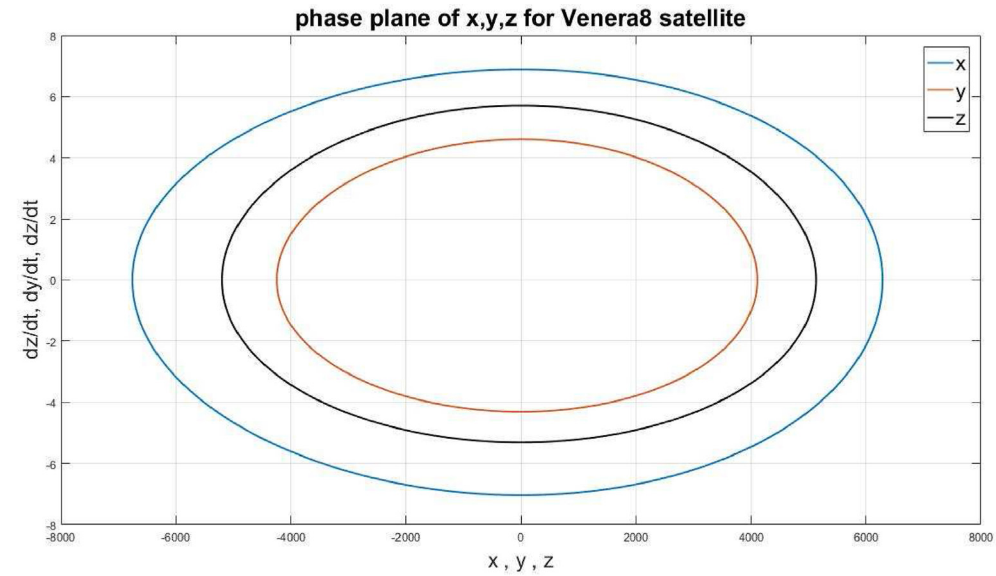
The scale factors of the transformation are

$h_1 = 1, h_2 = \rho, h_3 = 1$

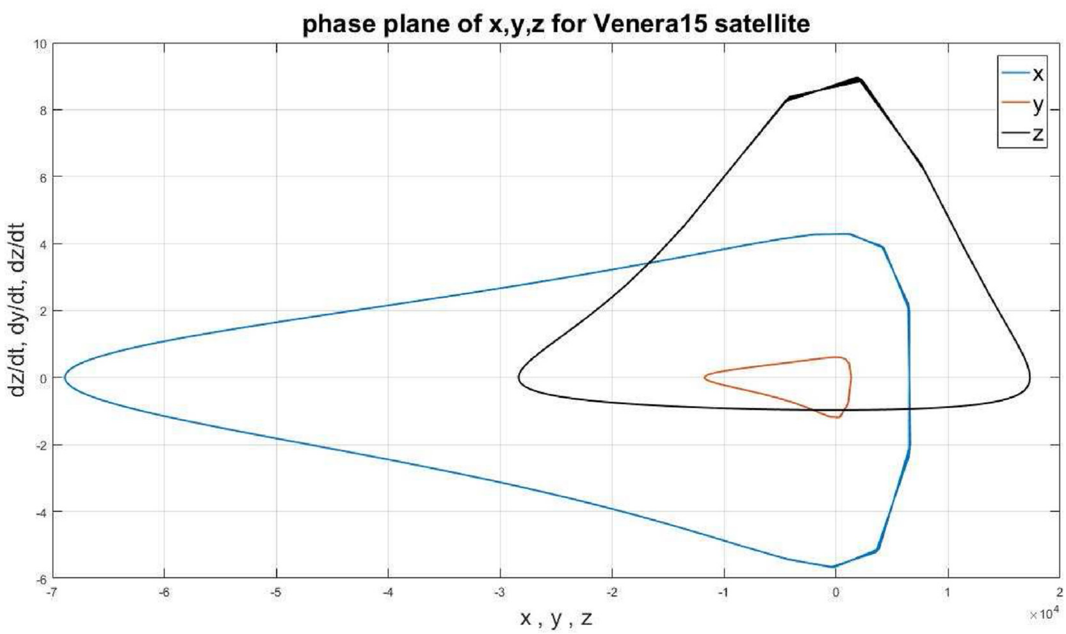
we have $\rho = (x^2 + y^2)^{1/2}$, $\theta = arctan\frac{y}{x}$, $Z = z$,

and $\dot{\rho} = \frac{x\dot{x}+y\dot{y}}{(x^2+y^2)^{0.5}}, \dot{\theta} = \frac{x\dot{y}+y\dot{x}}{(x^2+y^2)}, \dot{Z} = \dot{z}$

from the above equations: after some little reductions, we obtain



(a)



(b)

Fig. 8. The phase space plane of different Venus's satellites.

$$\dot{\rho}=\rho\left(1-\rho^2\right),\dot{\theta}=\frac{h}{(r)^2},\dot{Z}=\dot{z}$$

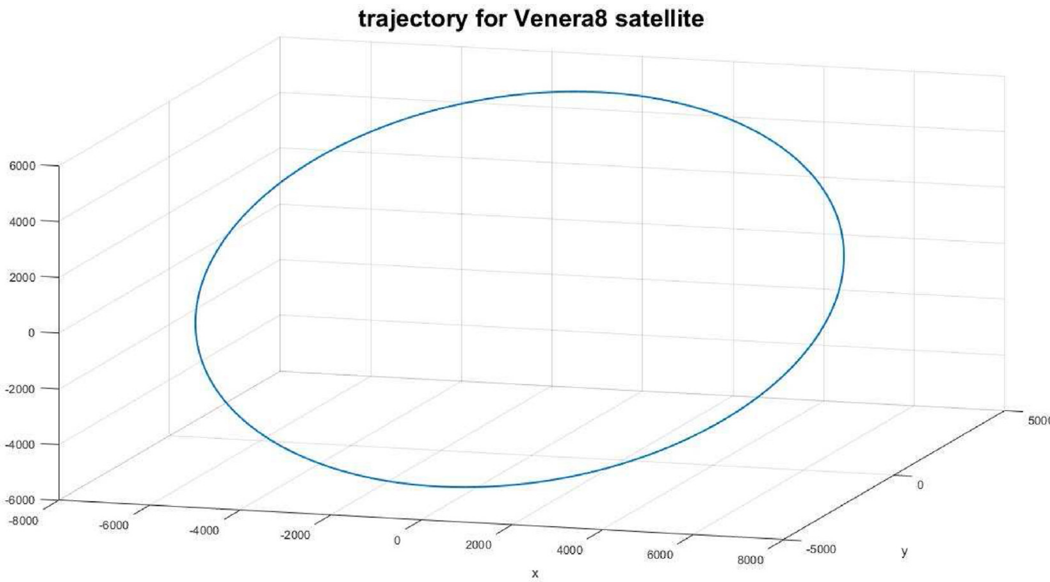
$$\ddot{\rho}=\rho(\dot{\theta})^2+2\left(\frac{\partial V}{\partial \rho}\right),\ddot{\theta}=\frac{-2(\dot{\rho})(\dot{\theta})}{\rho}+\frac{2}{(\rho)^2(\dot{\theta})}\frac{\partial V}{\partial \theta},\ddot{Z}=\frac{\partial V}{\partial Z}$$

the partial derivatives of the potential $\frac{\partial V}{\partial \rho}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial Z}$ can be given as

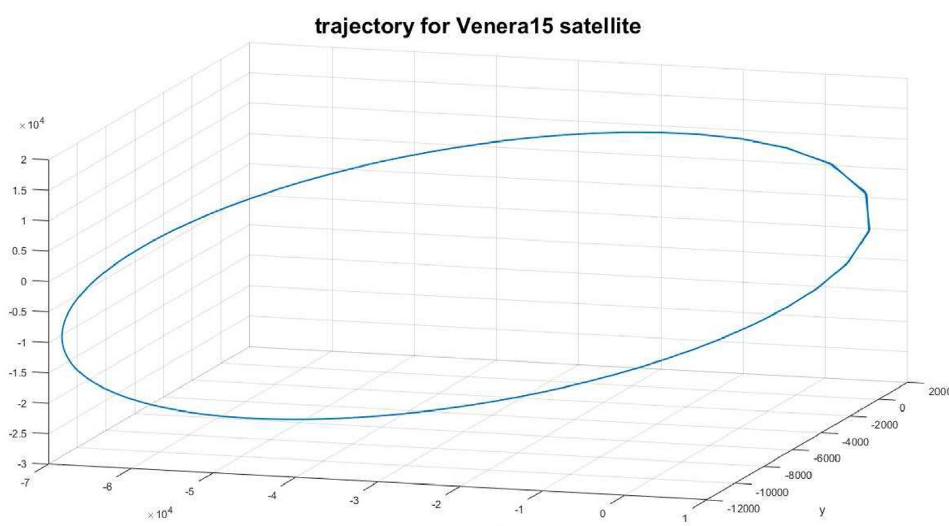
$$\frac{\partial V}{\partial \rho}=\cos\theta\frac{\partial V}{\partial x}+\sin\theta\frac{\partial V}{\partial y}$$

$$\frac{\partial V}{\partial \theta}=-\rho(\dot{\theta})cos\,\theta\frac{\partial V}{\partial x}+\rho(\dot{\theta})cos\,\theta\frac{\partial V}{\partial y} \tag{7}$$

$$\frac{\partial V}{\partial Z}=\frac{\partial V}{\partial z}$$



(a)



(b)

Fig. 9. The trajectory in space of the different Venus's satellites.

3.2. General equations of motion in terms of cylindrical coordinates

Let $(\rho, \theta, z) = (u_1, u_2, u_3)$, then the system of differential equations can be written as

$$\dot{u}_1 = u_4 = \dot{\rho}$$

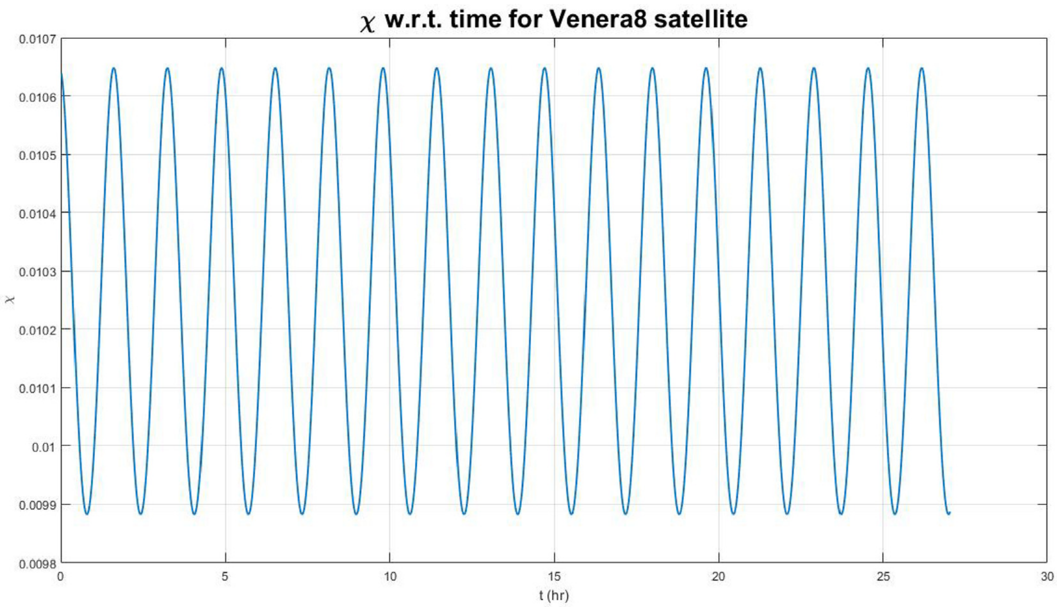
$$\dot{u}_2 = u_5 = \dot{\theta}$$

$$\dot{u}_3 = u_6 = \dot{z}$$

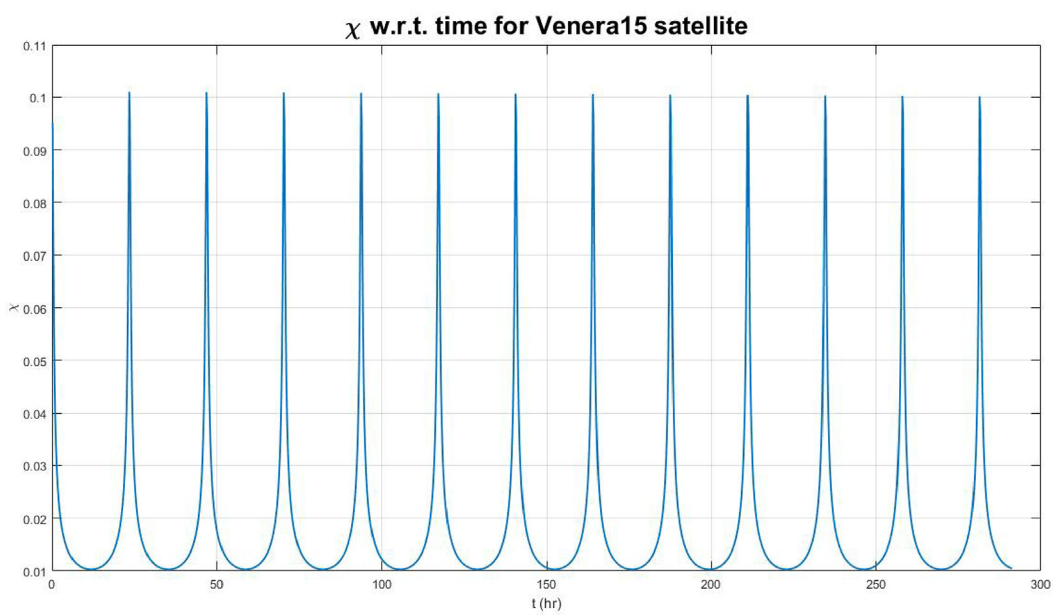
$$\ddot{u}_1 = \dot{u}_4 = u_1 u_5^2 + 2\frac{\partial v}{\partial u_1} \tag{8}$$

$$\ddot{u}_2 = \dot{u}_5 = \frac{-2u_4u_5}{u_1} + \frac{2}{u_1^2}\frac{\partial v}{\partial u_2}$$

$$\ddot{u}_3 = \dot{u}_6 = \frac{\partial v}{\partial u_3}$$



(a)



(b)

Fig. 10. The universal anomaly of Venus's satellites as a function of time.

4. The gravity perturbed trajectories

The potential V of the current problem including the zonal harmonics J_2 and J_3, besides the equatorial ellipticity coefficient C_{22} can be written as

$$V(x,y,z)=\frac{GM_v}{r}+\frac{GM_v}{r}\left(\frac{R_v}{r}\right)^2\left[J_2\left(\frac{1}{2}-\frac{3z^2}{2r^2}\right)\right]+\frac{GM_v}{r}\left(\frac{R_v}{r}\right)^2\left[3C_{22}\left(\frac{x^2-y^2}{r^2}\right)\right]+\frac{GM_v}{r}\left(\frac{R_v}{r}\right)^3\left[J_3\frac{z}{2r}\left(3-\frac{5z^2}{r^2}\right)\right] \tag{9}$$

where $\mu = GM$ the gravitational parameter of the planet, R its radius, and $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ the distance of the satellite with respect to the planet. The set of differential equations that describe the rate of change of the position and velocity of the satellite about the planet can be written as

$$\ddot{x} = \frac{\partial V}{\partial x}$$

$$\ddot{y} = \frac{\partial V}{\partial y} \tag{10}$$

$$\ddot{z} = \frac{\partial V}{\partial y}$$

where the planet's equatorial plane is chosen as the reference coordinate system.

4.1. Initial value algorithm

Now, using cylindrical coordinates, proceed to develop a general procedure for the final state predictions of the J_2, J_3, and C_{22} perturbed motion (Table 5). The computational steps of this algorithm are described as follows:

Input:

(1) x_o, y_o , z_o and $\dot{x}_o$, $\dot{y}_o$, $\dot{z}_o$ at $t = t_o$
(2) Flight time $t = t_f$
(3) Compute $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, and $\frac{\partial V}{\partial z}$

Output:

(1) u_i; $\dot{u}_i$, $i = 1, 2, 3$ at $t = t_f$
(2) x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$ at $t = t_f$

Computational steps:

(1) Find the analytical expressions of the partial derivatives by Equation (7).
(2) Compute the initial conditions $u_{oi}, i = 1, 2, ..., 6$ for the differential system of equation (8) by applying the transformation: $(x, y, z) \Rightarrow (x_o, y_o, z_o)$ and $(\dot{x}, \dot{y}, \dot{z}) \Rightarrow (\dot{x}_o, \dot{y}_o, \dot{z}_o)$
(3) Use these initial conditions to solve numerically the differential system of Equation (8) for $u_i; i = 1, 2, ..., 6$ at $t = t_f$
(4) Use $u_i, \dot{u}_i, i = 1, 2, 3$ to compute (x, y, z) and $(\dot{x}, \dot{y}, \dot{z})$ at $t = t_f$.
(5) End.

5. Results and discussions

In the present work, we have addressed numerically the two-body initial value problem. To

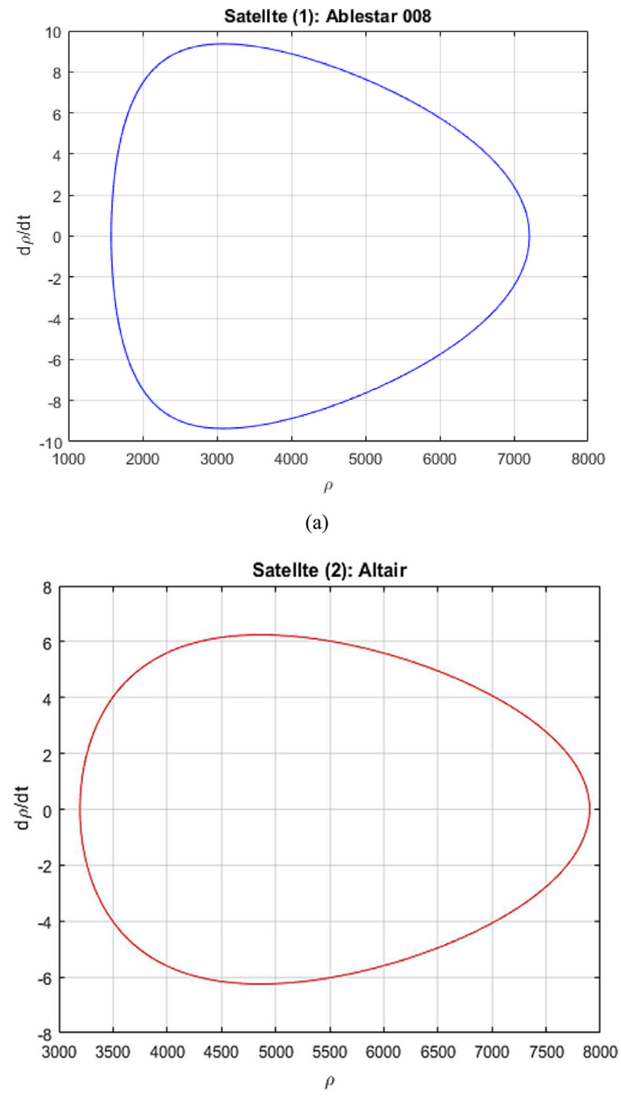


Fig. 11. The phase space portrait for the Earth's satellites in cylindrical coordinates.

overcome the mentioned inconveniences, the problem was expressed in terms of special 'universal' variables that regularize the equation of motion of the two-body. For some selected artificial satellite missions, we performed several numerical simulations using a homemade software package by Matlab. To show the validity of the universal variable approach to deal with various types of orbits, to do this, let us consider some different types of space missions, such as nearly circular, elliptic, and highly elliptic. We considered five artificial satellite missions around the two planets Earth and Venus (see Tables 1—4). Given the initial state vector $(\overline{r}, \overline{v})$ at a given initial epoch and numerically integrating the equations of the current problem, we obtain the

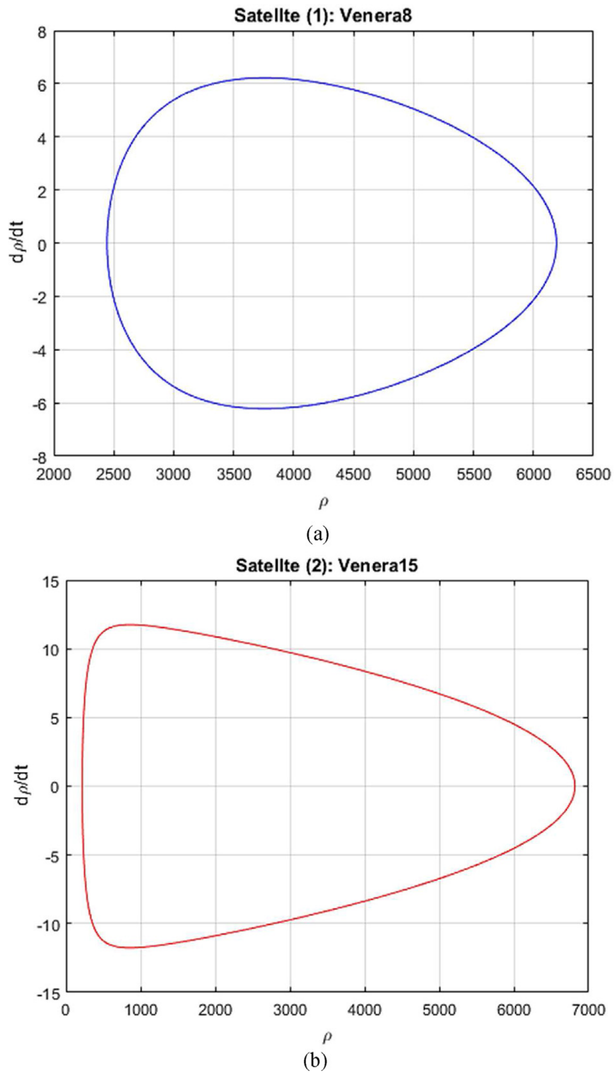


Fig. 12. The phase space portrait for the Venus's satellites in cylindrical coordinates.

components of position and velocity of the body as a function of time: $x(t), y(t), z(t), \dot{x}(t), \dot{y}(t),$ and $\dot{z}(t)$ (see Figs. 1, 2, 6 and 7). We note from the figures that the variation of the different variables is uniform so that the universal variable approach works equally well for the various types of orbits.

Fig. 3 depicts the phase space portrait for the selected artificial satellite missions around the Earth, while Fig. 8 depicts the phase space portrait for missions around the planet Venus. The phase space is a useful graphical method for determining qualitative information about the behavior of dynamic systems. The phase space of the current problem is described for the initial state vectors mentioned in Tables 1 and 3 It is straightforward to observe that most of the phase trajectories are almost elliptical closed orbits, while in the case of satellite Venera 15, Fig. 8b, the phase trajectories show a distortion introduced by its high eccentricity. Figs. 4 and 9 depict the trajectories of the regularized motion equations in the three-dimensional space of the initial-value problem. The two figures represent the trajectories of the selected missions around both the Earth and Venus. The initial state vectors $\overline{r}_0$ and $\overline{v}_0$ are given in Tables 1 and 3, as mentioned above. It is clear that, using the universal anomaly approach, all the three-dimensional space trajectories are sufficiently smooth for the different selected types of orbits. Figs. 5 and 10 show the smooth variations, for the various types of missions, of the universal anomaly as a function of time. Figs. 11 and 12 show the phase space portraits for the selected missions, in cylindrical coordinates, around Earth and the planet Venus, respectively. It is visible that the phase spaces are smooth. As a final result we can see from the current discussion that the universal anomaly approach facilitates the numerical and analytical treatments of the two-body dynamics, in particular the study of the artificial Earth satellite orbits.

Table 5. Earth and Venus parameters.

	Earth	Venus
μ (km^3·s^{-2})	398600.440	0.32486×10^6
M (Kg)	5.974×10^{24}	4.87×10^{24}
R (km)	6378.135	6052
J_2	$1.08261557 \times 10^{-3}$	4.4044×10^{-6}
J_3	-2.5327×10^{-6}	-2.1082×10^{-6}
C_{22}	$0.1574536043 \times 10^{-5}$	-2.2297×10^{-5}

5.1. Conclusions

The solution for the initial-value problem sometimes gives undesirable and inaccurate results due to the poor selection of variables that describe the dynamic problem. Also, the presence of singularities causes a lot of troubles in the final state prediction.

To overcome the mentioned problems, Sundmann introduced a family of transcendental functions to make all applicable general formulae valid for different types of orbits. We treated the problem numerically, using these transcendental functions, for some selected space missions and showed the appropriateness and validity of the universal anomaly approach to obtain better results.

Conflicts of interest

There are no conflicts of interest.

Acknowledgements

First, to the author expresses his deepest thanks to 'ALLAH' for continuous giving and helping to finish this work. A special gratitude to **Prof. Dr. Mohamed Radwan** for his complete inspiration through endless hours of scientific education supervision, suggestions concerning the problem, help and warm encouragement.

References

[1] Hou X, Xin X. A note on the full two-body problem and related restricted full three-body problem. Astrodynamics 2018;2:39−52.
[2] Sharaf MA, Dwidar HR. Initial value problem of space dynamics in universal Stumpff anomaly. Astrophys Space Sci 2018;363. 1−1.
[3] Everhart E, Pitkin ET. Universal variables in the two-body problem. Am J Phys 1983;51:712−7.
[4] Stiefel EL, Scheifele G. Linear and regular celestial mechanics: perturbed two-body motion, numerical methods, canonical theory. Berlin: Springer; 1971.
[5] Sharaf MA, Saad AS, Fadak MA. An efficient computational algorithm for the initial value problem associated with the universal Y functions. Astron Nachr 2016;337:235−8.
[6] Stumpff K. Neue formeln und hilfstafeln zur ephemeridenrechnung. Astron Nachr 1947;275:108−28.
[7] Herrick S. Universal variables. Astron J 1965;70:309.
[8] Sperling H. Computation of Keplerian conic sections. ARS J 1961;31:660−1.
[9] Mortari D, Elipe A. Solving Kepler's equation using implicit functions. Celestial Mech Dyn Astron 2014;118. 1−1.
[10] Tokis JN. A solution of Kepler's Equation. Int J Astron Astrophys 2014;4:683.
[11] Wisdom J, Hernandez DM. A fast and accurate universal Kepler solver without Stumpff series. Mon Not Roy Astron Soc 2015;453:3015−23.
[12] Raposo-Pulido V, Peláez J. An efficient code to solve the Kepler Equation-Hyperbolic case. Astron Astrophys 2018; 619:A129.
[13] Battin RH. An introduction to the mathematics and methods of astrodynamics by the American Institute of Aeronautics and Astronautics. Printed in the United States of America. Aiaa; 1999.