A Numerical Approach to Solve the Two-body Initial-value Problem with a Universal Variable


Abstract

The solution of the two-body initial-value problem gives inaccurate final state predictions for the orbital motions of artificial satellites. This is due to the presence of singularities and the poor selection of variables. In the current study, we numerically investigated the initial-value problem using the universal anomaly approach. To clarify the problem under concern, we carried out several numerical examples using a homemade software package. We considered five space missions, around the two planets Earth and Venus, which represent circular, near-circular, and elliptic orbits. We showed that the universal anomaly approach facilitates the numerical and analytical treatments of the two-body dynamics and works equally well for different types of orbits. Moreover, we developed a computation algorithm to handle the perturbed problem in cylindrical coordinates for the initial value problem taking into consideration the geopotential of the two planets up to the third zonal harmonic $J_3$ and the tesseral coefficient $C_{22}$.

Keywords: Cylindrical coordinate, Initial value problem, Perturbation, Trajectories, Venus

1. Introduction

In classical mechanics, the two-body initial-value problem may be defined as: At an instant of time, we give certain suitable initial conditions for the involved quantities, such as the velocity and position, to predict the subsequent motion. The two-body problem describes the dynamics of two celestial objects in close proximity, abstractly viewed as point masses. The problem assumes that the two bodies interact only through their mutual gravitational potential, and all other forces are ignored. The problem deals with the orbital and rotational motion of two finite bodies [1].

Beyond just the field of astrodynamics, the two-body problem has broad applicability in numerous engineering and scientific fields. One example of such a system is an artificial satellite that rotates around the Earth. Under the mentioned assumptions, the problem is simple and represents the only integrable system in celestial mechanics. In the case of nonspherical mass distributions of either one of the bodies, the problem becomes non-integrable and can exhibit chaotic dynamics. Since the time of Newton and Kepler, the solution of the classical gravitational two-body problem has been completely obtained. However, due to the existence of a singularity and deficiency of choice of variables, inconveniences and even difficulties occur when using classical methods. The formulation of the equations of motion is different depending on whether the conic section is a hyperbola, a parabola, or an ellipse. Definite formulae are required to determine the position for any case. When dealing with the same orbit for a long time they give satisfactory results. However, the orbit may suffer qualitative and quantitative changes in the computations of interplanetary trajectories. On the other hand, the conventional equations of motion of space
dynamic are unstable in the Lyapunov sense. Thus, for the motion of spacecrafts, the solution of these equations gives inaccurate predictions. To avoid these inconveniences, several authors introduced successful ‘universal’ variables that regularize the equations of motion of dynamics and maintain the same form of the solution in all cases [2–5]. Due to the importance of the current problem, it has been studied for decades. During last decades, studies were performed by several authors, they used numerical or approximation methods for solution of the Kepler’s equation. The earliest paper by Stumpff [6] and Herrick introduced a different formulation through the manipulation of series expansion of conventional conic variables [7]. Sperling obtained a related set of universal variables from the transformed equations of motion [8].

Mortari and Elipe [9] proposed a new approach to solve Kepler’s equation based on the use of implicit functions. The proposed method, with limited computational capability, is particularly suitable for space-based applications. Tokis [10] obtained a solution of the universal Kepler’s equation in closed form with the help of the two-dimensional Laplace technique, expressing the universal functions as a function of the universal anomaly and time. Wisdom and Hernandez [11] derived and present a fast and accurate solution of the initial value problem for Keplerian motion in universal variables that does

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Fig. 1. The position of the different Earth’s satellites x(t), y(t) and z(t) as a function of time.
not use the Stumpff series. They found that it performs better than methods based on the Stumpff series. Recently, Pulido and Pelaez [12] introduced a new approach for solving the Kepler equation for hyperbolic orbits. The authors tried to substantially improve well-known classic schemes based on the excellent properties of the Newton–Raphson iterative methods. The developed code provides fast and accurate solutions. Sharaf and Dwidar [2] set up the initial value problem of space dynamics in universal Stumpff anomaly and developed an analytical and computational approach. In the present study, we addressed numerically the two-body initial-value problem using the universal anomaly approach. We performed several numerical simulations to clarify the problem.

2. Universal formulation

The classical equations of the two-body problem cause troubles when a transition from one kind of orbit to another occurs. Therefore, to make the initial-value problem free from such troubles the relevant two-body equations are derived utilizing universal functions. To avoid using multiple formulations to describe motion in different orbits, the problem is generalized using transcendental
functions. Utilizing these functions, applicable general formulae can be obtained that are valid at the same time for all types of conic sections.

Now the key differential relationships can be summarized as follows [13]:

$$\frac{df}{dt} = \frac{p}{r} \left( \tan \frac{f}{2} \right) \frac{b \, dE}{dE} \frac{h}{r^2} \, dt$$

Because, for the three types of conic sections, we have

$$\frac{h}{p} = \sqrt{\mu} h = \sqrt{\mu} h = \sqrt{\mu}$$

then we can write

$$\sqrt{\mu} \, dt = r \left\{ \frac{\left( \sqrt{\mu} \, \tan \frac{f}{2} \right)}{d \left( \sqrt{\mu} \, E \right)} \frac{d \left( \sqrt{\mu} \, E \right)}{d \left( \sqrt{\mu} \, H \right)} \right\} = r \, dx$$

where $f$ is the true anomaly, $E$ the eccentric anomaly, $\mu$ the gravitational parameter, $H$ the hyperbolic anomaly, $a$ the semi-major axis, $p$ the semi-latus rectum, $h$ the angular momentum, $r$ the radius vector, and $b$ is the semiminor axis. The variable $\chi$ can be considered as a new and independent variable, (i.e) a kind of generalized anomaly. Therefore, the nonlinear equations of motion can be transformed into linear differential equations with constant coefficients, when $\chi$ is used as an independent variable instead of time. The transformation defined by

$$\sqrt{\mu} \, \frac{dt}{d\chi} = r$$

is called the Sundman transformation. Now the values $r, r, \sigma = \frac{dr}{d\chi}$ and $t$ can all be found as solutions of simple ordinary differential equations.

To do that, let us differentiate the identity

$$r^2 = r \cdot r$$

and obtain

$$\frac{r \, dr}{d\chi} = \frac{dt}{d\chi} \frac{dr}{dt} = r \sigma$$

From Lagrangian coefficients

$$\sigma = \frac{r \cdot v}{\sqrt{\mu}} = \sqrt{\mu} \tan \frac{1}{2} f$$

by cancelling the factor $r$ and differentiating the above equation, we have

$$\frac{d^2 r}{d\chi^2} = \frac{d \sigma}{d\chi} = r \frac{d}{d\chi} \left( r \cdot v \right) = r \left( \frac{2 \mu - \mu}{r - a} \right) = 1 - \frac{r}{a}$$

It is convenient here and subsequently to write $\alpha$ for the reciprocal of $a$, we have

$$\alpha = \frac{1}{a} = \frac{2 - \frac{v^2}{\mu}}{r}$$

and may be positive, negative, or zero. In summary, then

![Fig. 4. The trajectory in space of the different Earth’s satellites.](image-url)
\[
\frac{dr}{d\chi} = \sigma = \sqrt{\mu} \frac{d^2 t}{d\chi^2}
\]

\[
\frac{d^2 r}{d\chi^2} = \frac{d\sigma}{d\chi} = 1 - \alpha r
\]

\[
\frac{d^2 r}{d\chi^2} - \frac{d^2 \sigma}{d\chi^2} = \sqrt{\mu} \frac{d^4 t}{d\chi^4} = -\alpha \sqrt{\mu} \frac{d^2 t}{d\chi^2}
\]

so that \( \sigma, r, \) and \( t \) are solutions of the equations

\[
\frac{d^2 \sigma}{d\chi^2} + \alpha \sigma = 0, \quad \frac{d^2 r}{d\chi^2} + \alpha \frac{dr}{d\chi} = 0, \quad \frac{d^4 t}{d\chi^4} + \alpha \frac{d^2 t}{d\chi^2} = 0
\]  

(4)

The derivatives of the position vector \( r \) are given by

\[
\frac{dr}{d\chi} = \sqrt{\mu} v, \quad \frac{d^2 r}{d\chi^2} = \frac{\sigma}{\sqrt{\mu}} v - \frac{1}{r}
\]

lead in a similar manner to

\[
\frac{d^2 r}{d\chi^2} + \alpha \frac{dr}{d\chi} = 0
\]  

(5)

Equations (4) and (5) represent a set of linear differential equations with constant coefficients, for which solutions can be found without difficulty. Nevertheless, it is useful in this case to construct the solutions in a form using a family of special universal functions.

### 2.1. The universal functions \( U_n(\chi; \alpha) \)

These functions are first introduced by [2] as

\[
U_n(\chi; \alpha) = \chi^n \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha \chi^2)^k}{(n + 2k)!}
\]

Then the general form of Kepler equation was obtained as

\[
\sqrt{\mu}(t - t_0) = r_0 U_1(\chi; \alpha) + \sigma_0 U_2(\chi; \alpha) + U_3(\chi; \alpha)
\]

together with

\[
r = r_0 U_0(\chi; \alpha) + \sigma_0 U_1(\chi; \alpha) + U_2(\chi; \alpha)
\]

\[
\sigma = \sigma_0 U_0(\chi; \alpha) + (1 - \sigma_0) U_1(\chi; \alpha)
\]

The expressions for the Lagrangian coefficients are given as

\[
F = 1 - \frac{1}{r_0} U_2(\chi; \alpha) \quad G = \frac{r_0}{\sqrt{\mu}} U_1(\chi; \alpha) + \frac{\sigma_0}{\sqrt{\mu}} U_2(\chi; \alpha)
\]

\[
F_i = -\frac{\sqrt{\mu}}{rr_0} U_1(\chi; \alpha) \quad G_i = 1 - \frac{1}{r} U_2(\chi; \alpha)
\]  

(6)

We get the position and velocity of the satellite

\[
r = Fr_0 + Gv_0
\]

\[
v = Fv_0 + Gv
\]

The above equation (6) are termed ‘universal’ since it is void of singularities and valid for any conic sections.
3. Analytical formulation of cylindrical coordinates

In this section, the two-body initial-value problem of will be formulated utilizing cylindrical coordinates. We will develop a computation algorithm for the initial value problem of $J_2$, $J_3$, and $C_{22}$ gravity perturbed trajectories. Some numerical applications are carried out, for selected test orbits, for the problem of final state prediction.

3.1. Coordinate, velocity transformations

Let the rectangular coordinates $(x, y, z)$ of any point be expressed as a function of the cylindrical coordinate $(\rho, \theta, Z)$ so that

Fig. 6. The position of the different Venus's satellites $x(t)$, $y(t)$ and $z(t)$ as a function of time.
The scale factors of the transformation are

\[ h_1 = 1, \quad h_2 = \rho, \quad h_3 = 1 \]

we have \( \rho = \sqrt{x^2 + y^2} \) , \( \theta = \arctan \frac{y}{x} \), \( Z = z \),

and \( \dot{\rho} = \frac{xy + y\dot{y}}{(x^2 + y^2)^{3/2}} \), \( \dot{\theta} = \frac{x\dot{y} + y\dot{x}}{(x^2 + y^2)^{3/2}} \), \( \dot{Z} = \dot{z} \).

from the above equations: after some little reductions, we obtain

\[ x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = Z, \]

where

\[ 0 \leq \rho < \infty, \quad -\pi < \theta \leq \pi, \quad -\infty < Z < \infty \]

The scale factors of the transformation are
\[ \dot{\rho} = \rho (1 - \rho^2), \quad \dot{\theta} = \frac{h}{r^2}, \quad \dot{\bar{z}} = \dot{\bar{z}} \]

\[ \ddot{\rho} = \rho (\dot{\theta})^2 + 2 \left( \frac{\partial V}{\partial \rho} \right), \quad \ddot{\theta} = -2(\dot{\rho}) \frac{\partial V}{\rho} + \frac{2}{(\rho)^2} \frac{\partial V}{\partial \theta} \]

\[ \ddot{\bar{z}} = \frac{\partial V}{\partial \bar{z}} \]

The partial derivatives of the potential \( \frac{\partial V}{\partial \rho}, \frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial \bar{z}} \) can be given as

\[ \frac{\partial V}{\partial \rho} = \cos \theta \frac{\partial V}{\partial x} + \sin \theta \frac{\partial V}{\partial y} \]

\[ \frac{\partial V}{\partial \theta} = -\rho(\dot{\theta}) \cos \theta \frac{\partial V}{\partial x} + \rho(\dot{\theta}) \cos \theta \frac{\partial V}{\partial y} \]

\[ \frac{\partial V}{\partial \bar{z}} = \frac{\partial V}{\partial \bar{z}} \]
3.2. General equations of motion in terms of cylindrical coordinates

Let \((\rho, \theta, z) = (u_1, u_2, u_3)\), then the system of differential equations can be written as

\[
\begin{align*}
\dot{u}_1 &= u_4 = u_1 u_2^2 + 2 \frac{\partial v}{\partial u_1} \\
\dot{u}_2 &= u_5 = -\frac{2u_4 u_5}{u_1} + 2 \frac{\partial v}{u_1^2 \partial u_2} \\
\dot{u}_3 &= u_6 = \frac{\partial v}{\partial u_3}
\end{align*}
\]

Fig. 9. The trajectory in space of the different Venus’s satellites.
4. The gravity perturbed trajectories

The potential $V$ of the current problem including the zonal harmonics $J_2$ and $J_3$, besides the equatorial ellipticity coefficient $C_{22}$ can be written as

$$ V(x, y, z) = \frac{GM}{r} + \frac{GM}{r} \left( \frac{R}{r} \right)^2 \left[ J_2 \left( \frac{1}{2} \frac{3z^2}{r^2} \right) \right] + \frac{GM}{r} \left( \frac{R}{r} \right) \left[ J_3 \frac{z}{2r} \left( 3 - \frac{5z^2}{r^2} \right) \right] $$
where \( \mu = GM \) the gravitational parameter of the planet, \( R \) its radius, and \( r = (x^2 + y^2 + z^2)^{1/2} \) the distance of the satellite with respect to the planet. The set of differential equations that describe the rate of change of the position and velocity of the satellite about the planet can be written as

\[
\begin{align*}
\dot{x} &= \frac{\partial V}{\partial x} \\
\dot{y} &= \frac{\partial V}{\partial y} \\
\dot{z} &= \frac{\partial V}{\partial y}
\end{align*}
\]

(10)

where the planet’s equatorial plane is chosen as the reference coordinate system.

4.1. Initial value algorithm

Now, using cylindrical coordinates, proceed to develop a general procedure for the final state predictions of the \( J_2, J_3, \) and \( C_{22} \) perturbed motion (Table 5). The computational steps of this algorithm are described as follows:

Input:

1. \( xo, yo, zo \) and \( \dot{x}_0, \dot{y}_0, \dot{z}_0 \) at \( t = t_0 \)
2. Flight time \( t = t_f \)
3. Compute \( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \) and \( \frac{\partial V}{\partial z} \)

Output:

1. \( u_i, \dot{u}_i, i = 1, 2, 3 \) at \( t = t_f \)
2. \( x, y, z \) and \( \dot{x}, \dot{y}, \dot{z} \) at \( t = t_f \)

Computational steps:

1. Find the analytical expressions of the partial derivatives by Equation (7).
2. Compute the initial conditions \( u_{oi}, i = 1, 2, \ldots, 6 \) for the differential system of equation (8) by applying the transformation: \( (x, y, z) \Rightarrow (x_0, y_0, z_0) \) and \( (\dot{x}, \dot{y}, \dot{z}) \Rightarrow (\dot{x}_0, \dot{y}_0, \dot{z}_0) \)
3. Use these initial conditions to solve numerically the differential system of Equation (8) for \( u_i, i = 1, 2, \ldots, 6 \) at \( t = t_f \)
4. Use \( u_i, \dot{u}_i, i = 1, 2, 3 \) to compute \( (x, y, z) \) and \( (\dot{x}, \dot{y}, \dot{z}) \) at \( t = t_f \).
5. End.

5. Results and discussions

In the present work, we have addressed numerically the two-body initial value problem. To overcome the mentioned inconveniences, the problem was expressed in terms of special ‘universal’ variables that regularize the equation of motion of the two-body. For some selected artificial satellite missions, we performed several numerical simulations using a homemade software package by Matlab. To show the validity of the universal variable approach to deal with various types of orbits, to do this, let us consider some different types of space missions, such as nearly circular, elliptic, and highly elliptic. We considered five artificial satellite missions around the two planets Earth and Venus (see Tables 1–4). Given the initial state vector \((r, \mathbf{v})\) at a given initial epoch and numerically integrating the equations of the current problem, we obtain the...
components of position and velocity of the body as a function of time: \(x(t), y(t), z(t), \dot{x}(t), \dot{y}(t),\) and \(\dot{z}(t)\) (see Figs. 1, 2, 6 and 7). We note from the figures that the variation of the different variables is uniform so that the universal variable approach works equally well for the various types of orbits.

Fig. 3 depicts the phase space portrait for the selected artificial satellite missions around the Earth, while Fig. 8 depicts the phase space portrait for missions around the planet Venus. The phase space is a useful graphical method for determining qualitative information about the behavior of dynamic systems. The phase space of the current problem is described for the initial state vectors mentioned in Tables 1 and 3. It is straightforward to observe that most of the phase trajectories are almost elliptical closed orbits, while in the case of satellite Venera 15, Fig. 8b, the phase trajectories show a distortion introduced by its high eccentricity. Figs. 4 and 9 depict the trajectories of the regularized motion equations in the three-dimensional space of the initial-value problem. The two figures represent the trajectories of the selected missions around both the Earth and Venus. The initial state vectors \(\mathbf{r}_0\) and \(\mathbf{v}_0\) are given in Tables 1 and 3, as mentioned above. It is clear that, using the universal anomaly approach, all the three-dimensional space trajectories are sufficiently smooth for the different selected types of orbits. Figs. 5 and 10 show the smooth variations, for the various types of missions, of the universal anomaly as a function of time. Figs. 11 and 12 show the phase space portraits for the selected missions, in cylindrical coordinates, around Earth and the planet Venus, respectively. It is visible that the phase spaces are smooth. As a final result we can see from the current discussion that the universal anomaly approach facilitates the numerical and analytical treatments of the two-body dynamics, in particular the study of the artificial Earth satellite orbits.

5.1. Conclusions

The solution for the initial-value problem sometimes gives undesirable and inaccurate results due to the poor selection of variables that describe the dynamic problem. Also, the presence of singularities causes a lot of troubles in the final state prediction.

To overcome the mentioned problems, Sundmann introduced a family of transcendental functions to make all applicable general formulae valid for different types of orbits. We treated the problem numerically, using these transcendental functions, for some selected space missions and showed the appropriateness and validity of the universal anomaly approach to obtain better results.

Conflicts of interest

There are no conflicts of interest.

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