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ORIGINAL ARTICLE

Global Stability Analysis and Numerical Simulation for Nonlinear Ecological Model with Delay

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Abstract

Discrete models are particularly useful for modelling population dynamics when the population size remains small over several generations or when it is relatively constant within a single generation. We focus on finding effective solutions to the challenges posed by such populations. In our research, we have successfully used qualitative analytic techniques to study a three species model. It is important to consider the reproductive process and other population dynamics as happening in real-time, even for species with unclear reproductive seasons. While the delay technique has introduced some complexities, we have identified sufficient conditions to address them. Our study examines the global stability of a three species ecological model that does not consider delayed intraspecific competition. We analyze a delayed Lotka Volterra system, which demonstrates global stability when the interaction matrix is effective. We present numerical simulations to illustrate the theoretical results of the delay differential equation. Since delay differential equation models are always challenging to solve, we propose the use of JiTCDDE (just-in-time compilation for delay differential equations) of the DDE integration method to solve the dynamical three species models.

Mathematics Subject Classification (2023): 35A01, 92C17, 93D301

Keywords: Equilibrium points, Global stability, Time delay, Numerical simulation

1. Introduction

Ecology is a branch of science that investigates how living things connect with their surroundings. Understanding the variables that influence the distribution, abundance, and interactions of organisms in their natural habitats is the primary objective of ecological study. In population dynamics, disease modelling, neural networks, and other significant fields of science, delay differential equations (DDEs) have been extensively used to describe these interactions.

When accounting for time delays that are present in biological systems, such as the maturation period of biological species, the time required for synaptic transmission between neurons, and the incubation period in epidemic models, it is frequently more accurate to use differential equations than it is to use ordinary differential equations (ODEs).

The interaction between predators and prey is one of the most vital biological relationships, and as such, researchers have given it significant attention [2−4,10,15−17]. To gain a better understanding of predator-prey dynamics, several mathematical models, including those that incorporate time delays in DDEs, have been proposed and developed. In the field of mathematical biology, the stability analysis of Lotka Volterra systems with delays [6,8,9,12,13,18] is of great interest because it sheds light on the long-term behavior of biological systems. The dynamics of prey-predator models have been extensively studied to examine the stability of positive equilibria and the presence of non-negative equilibria.

This study examines three species delayed Lotka Volterra systems, which are intricate and capable of displaying a variety of behaviors based on the initial conditions and parameter values. We concentrate on

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the system's difficult global stability analysis, which calls for the creation of cutting-edge mathematical methods. Exploring the system's dynamics and determining the prerequisites for its long-term security are our main objectives. We provide a comprehensive explanation of our mathematical model and assumptions in Section 2, followed by a proof of global stability with discrete delay in Section 3. We provide numerical models in Section 4 to demonstrate the viability of our key findings. In the final section, we offer a succinct summary of our results.

2. Mathematical model

We investigate the global stability of a three species Lotka Volterra system with delay

$$\left.\begin{aligned} x_1'(t)&=[a_1x_1(t)+a_2x_2(t-\tau_{12})+a_3x_3(t-\tau_{13})+\alpha]x_1(t),\\ x_2'(t)&=[b_2x_2(t)+b_1x_1(t-\tau_{21})+b_3x_3(t-\tau_{23})+\beta]x_2(t),\\ x_3'(t)&=[c_3x_3(t)+c_1x_1(t-\tau_{31})+c_2x_2(t-\tau_{32})+\gamma]x_3(t)\end{aligned}\right\}\tag{2.1}$$

with conditions

$$x_i=\psi_i(t)\geq 0,(i=1,2,3),t\in[-\tau_0,0],\psi_i(0)>0.$$

Where x_i are the species density i, and α,β,γ are the rates of generation, $\tau_{ij}>0,i\neq j,\{i,j=1,2,3\},\tau_0=\max\{\tau_{ij}\},a_i,b_i,c_i,\{i=1,2,3\}$ “are constants”, and $\psi_i(t)$ continuous on $[-\tau_0,0]$.

And (2.1) assumed to take a single non-negative equilibrium $\widehat{x}=(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)$. Sufficient conditions are provided for system (2.1) to guarantee the global stability of $\widehat{x}$. One can verify that the following condition (I) is stronger than any known necessary restrictions to guarantee the global stability of $\widehat{x}$ [14].

2.1. Condition (I)

$$\left.\begin{aligned} &a_1<0,b_2<0,c_3<0,-a_2b_1\leq a_1b_2,\\ &\text{if } a_2b_1<0,a_2b_1<a_1b_2,\\ &\text{if } -a_3c_1<0,-a_3c_1\leq a_1c_3,a_3c_1<a_1c_3,\\ &\text{if } b_3c_2<0,-b_3c_2\leq b_2c_3,b_3c_2<b_2c_3,\\ &b_2\leq b_1,a_1\leq c_2,a_3\leq b_2.\end{aligned}\right\}\tag{2.2}$$

In reality, it will be established that, in the particular case of, condition (I) is necessary for the system global stability.

$$a_1<0,b_2<0,-a_2b_1\leq a_1b_2,a_2b_1<a_1b_2 \text{ if } a_2b_1<0.$$

3. Global stability

Theorem 3.1. If and only if condition (I) is satisfied, (2.1) is globally stable when $\tau_{ij}\geq 0$.

Proof. Sufficiency. By using the transformation

$$\widehat{x}_1=x_1-\overline{x}_1,\widehat{x}_2=x_2-\overline{x}_2,\widehat{x}_3=x_3-\overline{x}_3\tag{3.1}$$

where, (2.1) has a non-negative equilibrium $\widehat{x}=(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)$, it becomes

$$\left.\begin{aligned} x_1'(t)&=[a_1x_1(t)+a_2x_2(t-\tau_{12})+a_3x_3(t-\tau_{13})](x_1(t)+\widehat{x}_1),\\ x_2'(t)&=[b_2x_2(t)+b_1x_1(t-\tau_{21})+b_3x_3(t-\tau_{23})](x_2(t)+\widehat{x}_2),\\ x_3'(t)&=[c_3x_3(t)+c_1x_1(t-\tau_{31})+c_2x_2(t-\tau_{32})](x_3(t)+\widehat{x}_3),\end{aligned}\right\}\tag{3.2}$$

we took $x_i(t)$ rather than $\overline{x}_i(t)$. Considering the Lyapunov function [5] $V:D(-\tau,0)^2\to\mathbb{R}$ with

$$V(\psi)=\sum_{i,j=1;i\neq j}^{3}e_{ij}\int_{-\tau_{ij}}^{0}\psi_j^2(\rho)\,d\rho+\sum_{i=1}^{3}d_i(\widehat{x}_i\,ln(\widehat{x}_i)\tag{3.3}$$
$$-\widehat{x}_i\,ln(\widehat{x}_i+\psi_i(0))+\psi_i(0)),$$

where

$$d_1=-2a_1^2b_1^2,d_2=-2a_1^2b_2^2,d_3=-2b_2^2c_3^2,$$
$$e_{12}=a_2^2b_1^2,e_{21}=a_1^2b_1^2,e_{13}=b_2^2a_3^2,$$
$$e_{31}=c_1^2a_3^2,e_{23}=c_2^2b_3^2,e_{32}=c_2^2b_2^2.$$

By condition (I),

$$\left.\begin{aligned}\dot{V}(\psi)\leq&-b_1^2[a_1\psi_1(0)+a_2\psi_2(-\tau_{12})+a_3\psi_3(-\tau_{13})]^2\\ &-c_2^2[b_1\psi_1(-\tau_{21})+b_2\psi_2(0)+b_3\psi_3(-\tau_{23})]^2\\ &-b_2^2[c_1\psi_1(-\tau_{31})+c_2\psi_2(-\tau_{32})+c_3\psi_3(0)]^2.\end{aligned}\right\}\tag{3.4}$$

Hence

$$\left.\begin{aligned}&F=\{\psi=(\psi_1,\psi_2,\psi_3):\dot{V}(\psi)=0\}=\{\psi=(\psi_1,\psi_2,\psi_3):a_1\psi_1(0)+a_2\psi_2(-\tau_{21})\\ &+a_3\psi_3(-\tau_{13})=0,\\ &b_1\psi_1(-\tau_{21})+b_2\psi_2(0)+b_3\psi_3(-\tau_{23})=0,\\ &c_1\psi_1(-\tau_{31})+c_2\psi_2(-\tau_{32})+c_3\psi_3(0)=0\}.\end{aligned}\right\}\tag{3.5}$$

We simply demonstrate the Lasalle's invariant set included in F offers only a zero solution [5] to demonstrate the global stability. Now Consider $x(t) = (x_1(t), x_2(t), x_3(t))$ is any solution in F therefore, it must be satisfying

$$\left.\begin{aligned} a_1x_1(t) + a_2x_2(t-\tau_{12}) + a_3x_3(t-\tau_{13}) = 0,\\ b_2x_2(t) + b_1x_1(t-\tau_{21}) + b_3x_3(t-\tau_{23}) = 0,\\ c_3x_3(t) + c_1x_1(t-\tau_{31}) + c_2x_2(t-\tau_{32}) = 0.\end{aligned}\right\} \tag{3.6}$$

Obviously, the equalities in (3.6) and system (3.2) leads to $x_i' = 0$. Hence, $\hat{x}$ is globally stable for (2.1). Necessity. Linearized system's (2.1) characteristic equation

$$\left.\begin{aligned} x_1' = a_1x_1(t) + a_2x_2(t-\tau_{12}) + a_3x_3(t-\tau_{13}),\\ x_2' = b_2x_2(t) + b_1x_1(t-\tau_{21}) + b_3x_3(t-\tau_{23}),\\ x_3' = c_3x_3(t) + c_1x_1(t-\tau_{31}) + c_2x_2(t-\tau_{32}),\end{aligned}\right\} \tag{3.7}$$

becomes

$$\lambda^3 + Ly^2 + M\lambda + N + Pe^{-\lambda\tau} = 0 \tag{3.8}$$

where

$$L = a_1\hat{x}_1 + b_2\hat{x}_2 + c_3\hat{x}_3,\ M = a_1b_2\hat{x}_1\hat{x}_2 + b_2c_3\hat{x}_2\hat{x}_3 + a_1c_3\hat{x}_1\hat{x}_3,\ N = a_1b_2c_3\hat{x}_1\hat{x}_2\hat{x}_3,$$

$$P = -(b_1c_1b_3\hat{x}_1\hat{x}_2\hat{x}_3 + a_3b_1c_2\hat{x}_1\hat{x}_2\hat{x}_3),\ \tau = \sum_{i,j=1, i\neq j}^{3} \tau_{ij}\ .$$

If $\tau = 0$, then (3.8) becomes

$$\lambda^3 + L\lambda^2 + M\lambda + (N+P) = 0 \tag{3.9}$$

The uniqueness of the positive equilibrium $\hat{x}$ reduces to $N+ P\neq 0$. Then (2.1) if $\tau_{ij} = 0$, $i\neq j, \{i,j = 1,2,3\}$, is globally stable, $N+ P\neq 0$, the eigenvalues of (3.9) possess a negative real parts,

$$L > 0, M = 0, N+P > 0 \tag{3.10}$$

We used the reality, $L = 0$ is the sufficient and necessary condition for (2.1), $\tau_{ij} = 0, i\neq j, \{i,j = 1,2,3\}$, to be integrable [7]. In the case of $b_1c_1b_3 < 0$ and $a_3b_1c_2 < 0$, if condition (I) fails, there exist a τ_0 therefore for τ beside τ_0, (2.1) can have a periodic solution. If condition (I) does not imply (3.10) in addition to $b_1c_1b_3 < 0$ and $a_3b_1c_2 < 0$ reduces $P^2 > N^2$.
Putting $x + iy = \lambda$ in (3.8), we get,

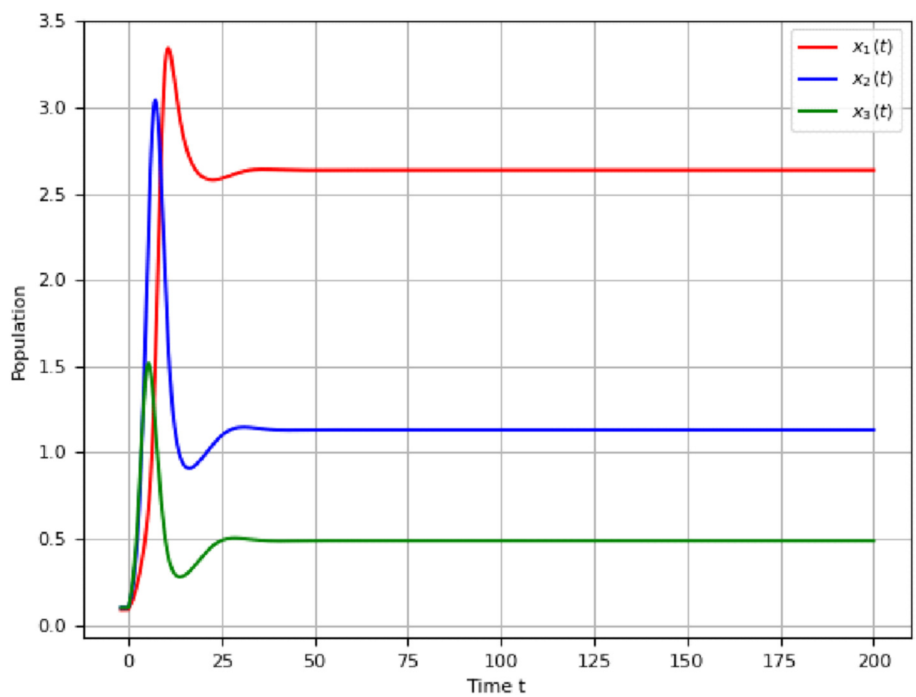

Fig. 1. Phase portrait and time series plot of the system (2.1) when $a_1 = -0.22, a_2 = 0.21, a_3 = -0.22, b_1 = -0.21$, $b_2 = -0.21$, $b_3 = 0.125$, $c_1 = -0.2, c_2 = -0.2, c_3 = -0.25, \alpha = 0.45, \beta = 0.73, \gamma = 0.875$.

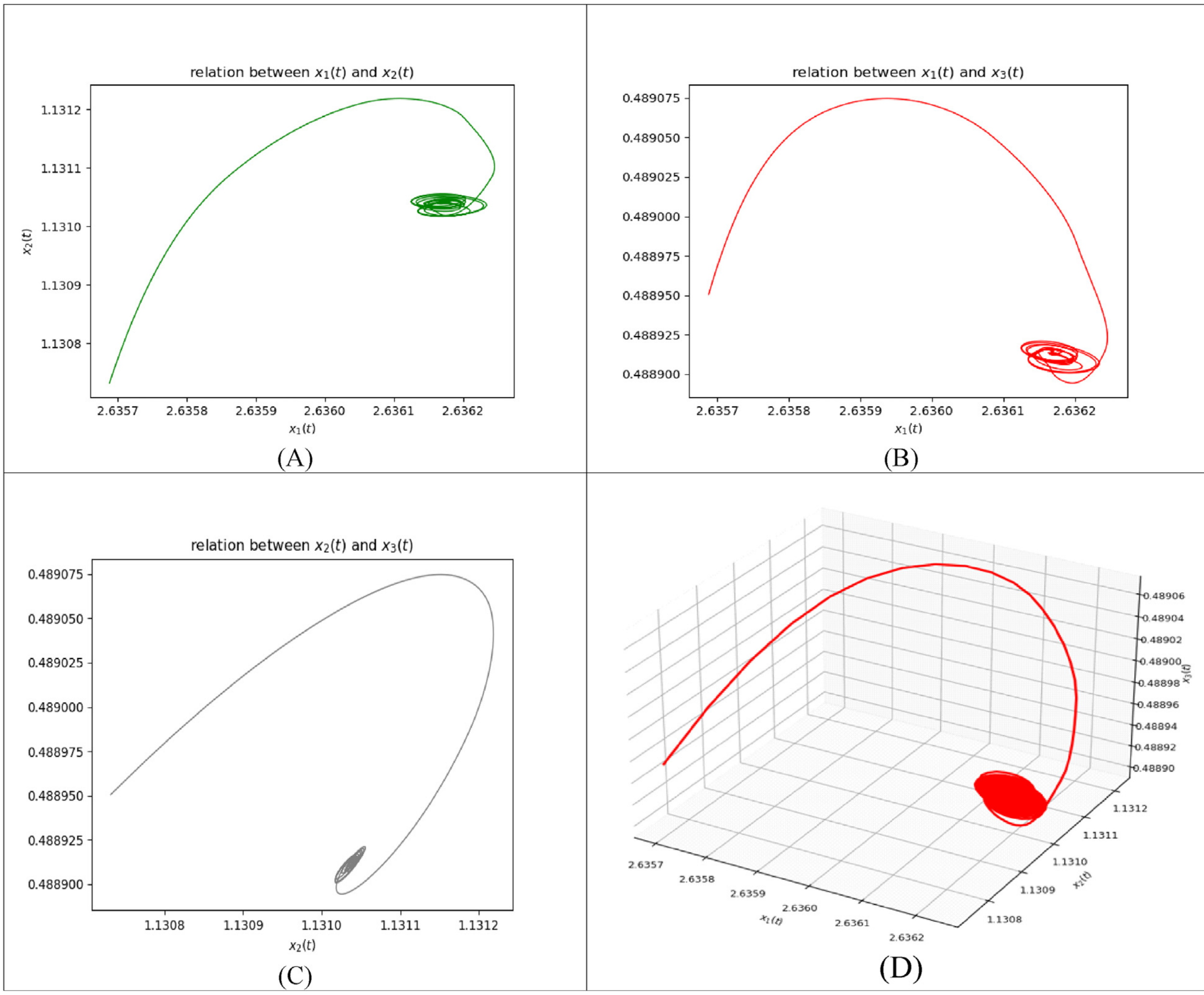

Fig. 2. Phase portrait and time series plot of the system (2.1) when $a_1 = -0.22, a_2 = 0.21, a_3 = -0.22, b_1 = -0.21, b_2 = -0.21, b_3 = 0.125, c_1 = -0.2, c_2 = -0.2, c_3 = -0.25, \alpha = 0.45, \beta = 0.73, \gamma = 0.875$.

$$\left.\begin{array}{l} x^3 + Lx^2 - y^2(3x+L) + M + N + Pe^{-\tau x}\cos(\tau y) = 0, \\ y^3 - y(3x^2 + 2Lx + M) + Pe^{-\tau x}\sin(\tau y) = 0. \end{array}\right\} \tag{3.11}$$

Putting $x = 0$, in (3.11) we get

$$Ly^2 - (M+N) = P\cos(\tau y) \tag{3.12}$$

$$y^3 - My = -P\sin \tau y \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$\left(Ly^2 - M - N\right)^2 + \left(My - y^3\right)^2 = P^2 \tag{3.14}$$

This implies that $P^2 > N^2$ is also true. Allowing y to be a non-negative solution for (3.14) and substituting in (3.13), we obtain τ_0 and (3.8) has an eigenvalue iy at τ_0 and $x' > 0$. (2.1) has a periodic solution beside τ_0, according to Hopf bifurcation theorem [5].

4. Numerical simulations

This section uses numerical simulation to enable us visualize the aforementioned analytical results and comprehend the impact of changing

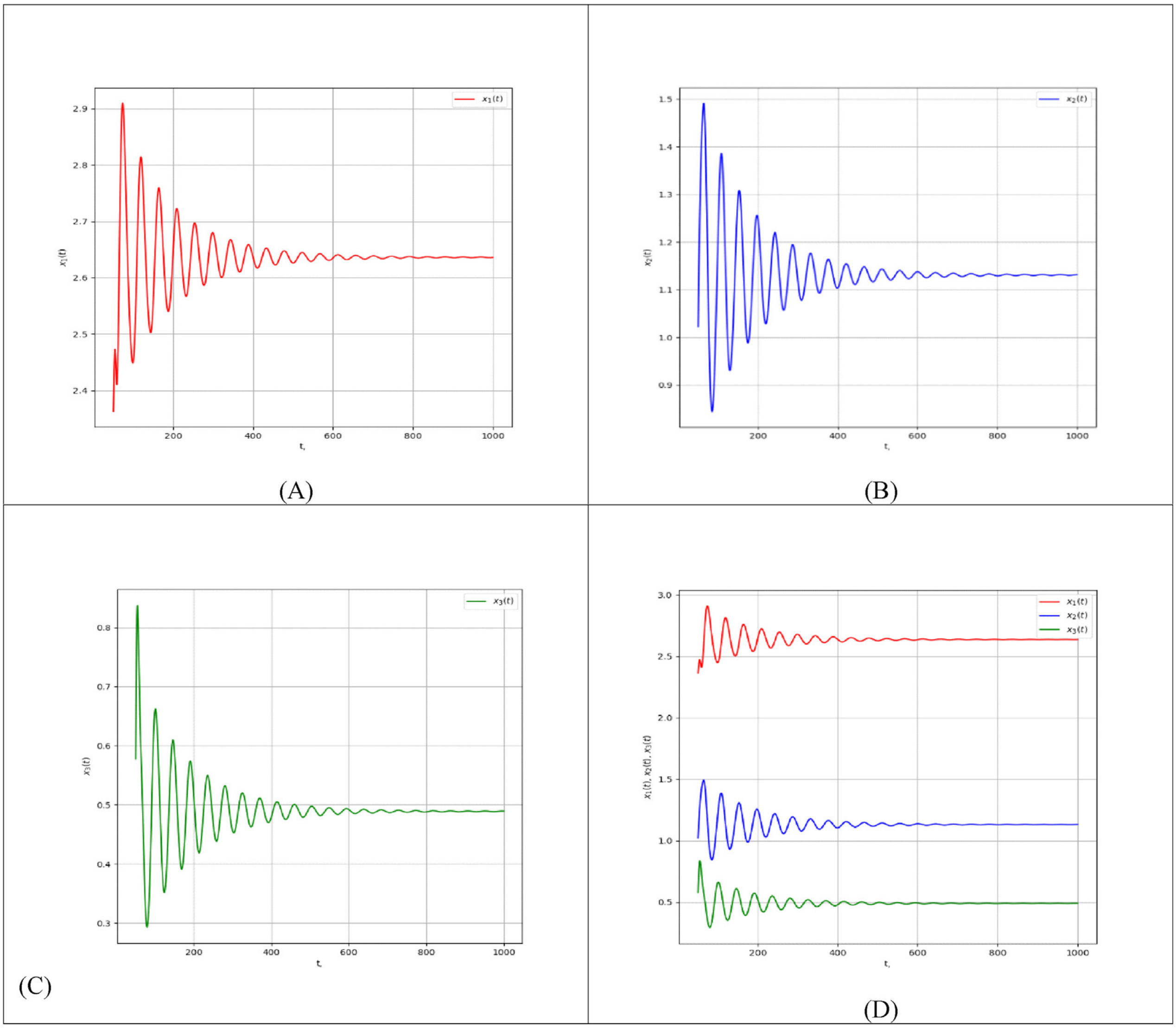

Fig. 3. Phase portrait and time series plot of the system (2.1) when $a_1 = -0.22, a_2 = 0.21, a_3 = -0.22, b_1 = -0.21$, $b_2 = -0.21$, $b_3 = 0.125$, $c_1 = -0.2, c_2 = -0.2, c_3 = -0.25, \alpha = 0.45, \beta = 0.73, \gamma = 0.875$.

parameters on the system's overall dynamics (2.1). The goals of this study are to validate the outcomes of our analytical work and to identify the set of control parameters that have an impact on the dynamics of the system. As a result, system (2.1) is numerically solved for various starting condition sets and parameter sets. As can be seen in the figures Figs. 1–5 below, system (2.1) has a globally asymptotically steady positive equilibrium point for the following set of artificial parameters. JiTCDDE, a Python program created by [1], was employed.

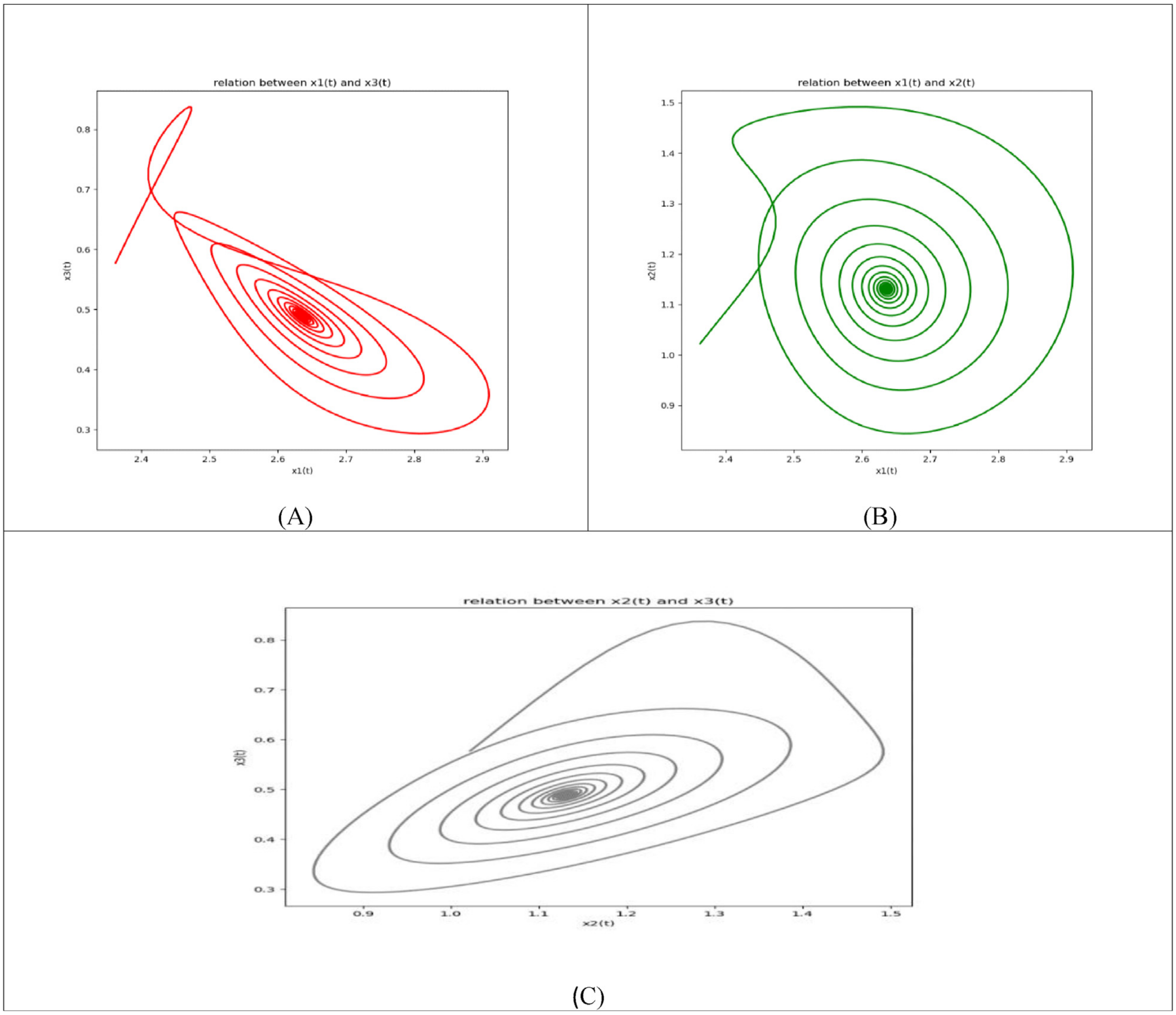

Fig. 4. Phase diagram for the behaviour of the model equation (2.1) for the global stability of population with parametric values taken as $a_1 = -0.22$, $a_2 = 0.21$, $a_3 = -0.22$, $b_1 = -0.21$, $b_2 = -0.21$, $b_3 = 0.125$, $c_1 = -0.2$, $c_2 = -0.2$, $c_3 = -0.25$, $\alpha = 0.45$, $\beta = 0.73$, $\gamma = 0.875$.

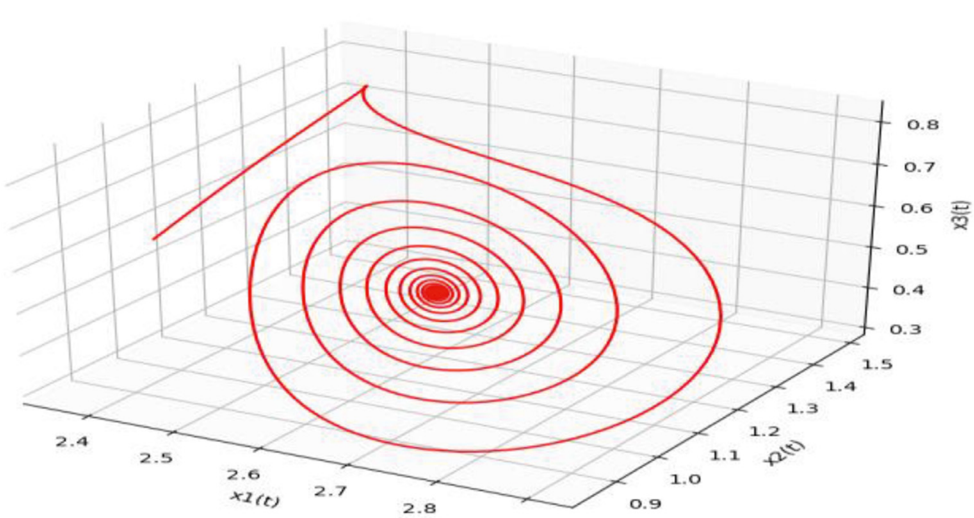

Fig. 5. Phase diagram for the periodic solutions of the model equation (2.1).

5. Conclusions

In this paper, we recognised a Lotka Volterra system with delay in three species. [11,14] was extended to the three species with discrete delays. We demonstrated that condition (I) is a sufficient and necessary condition for global stability delays in some ways. If condition (I) is studied, the appearance of a periodic solution is also demonstrated. By developing appropriate Lyapunov-function, we were successful in obtaining certain sufficient and essential conditions for the positive equilibrium's global stability. Our findings indicate that global stability of the positive equilibrium is still possible if and only if the system's matrix fulfils condition (I). Finally, numerical simulation is used with a hypothetical set of parameter values to complete our understanding of the global stability behaviour of system (2.1).

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Conflict of interest

There are no conflicts to declare.

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