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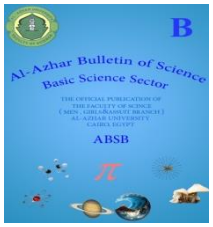

CORONAVIRUS EPIDEMIC MODEL WITH ISOLATION AND NONLINEAR INCIDENCE RATE

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ABSTRACT

In our article we proposed an epidemiological model consisting of five compartments that describe corona virus disease with isolation. We developed an SEIR system to produce the dynamical behaviour of infection by adding isolation compartment Φ(t) (that is because of the isolation of the infected individuals will reduce the spread of the disease). We formulated our model with nonlinear incidence rate. First, we discussed the positivity and boundedness of the model. Then we calculated the basic reproduction number of the model under certain conditions. By analysing the local and the global stability of our model, it is noticed that this stability depends on the basic reproductive number. We found that the system has no endemic equilibrium state if $R_0 < 1$. Finally, we discussed the global stability by using Lyapunov function of Goh-volterra type with LaSalle's invariance principle. It was shown that the system has a unique equilibrium point which is global asymptotically stable.

Keywords: COVID-19; Epidemiology; Reproduction number; Stability; Nonlinear incidence rate; Isolation.

1. INTRODUCTION

Researchers used the mathematical models to understand the behavior of an infection in any community and investigate the necessary conditions it will be wiped out or continued, to get insight into the spread of the diseases, and also to control the emerging and reemerging the disease. Different forms of epidemiological models are used to study and analyze the spread of disease for example, coronavirus, Influenza, HIV, SARS, and many other diseases.

Coronavirus is a large family of viruses which cause illness of the range between the common cold and sever acute respiratory syndrome [1]. It has been classified by the World Health Organization (WHO) as a globe pandemic [2]. This virus transmitted when the infected individuals cough, sneeze, or exhale due to close contact. The incubation period (infectious) of the coronavirus is about ten days is declared by National Health Commission of the People's Republic of China.

Recently, epidemics are simulated by different ways for example, SIS, SIR and SEIR models. Also those models have been used to describe the coronavirus disease. SEIR model is proposed and investigated by Cooke and Driessche [3], then it has become the most important model in controlling of diseases. For this, SEIR models are studied by using ODEs, PDEs and SDEs [4-7]. In [8] the authors formulated an SEIR epidemic model to describe coronavirus disease according to some control strategies such as quarantine. A mathematical model for Middle East respiratory syndrome coronavirus (MERS-COVID) transmission dynamics was used to estimate the transmission rates in two periods due to the implementation of intensive interventions [9-14].

It is obvious that human to human contact is the essential reason of spread the pandemic disease COVID-19 or coronavirus disease. Controlling diseases is very important public health to reduce the spread of the diseases. Infected persons can be treated by several control strategies for example isolation, vaccination, and the use of treatment. In this paper, we concentrate on isolation that applied to infected persons to prevent them from further contacts and transmission to other people.

A. Zeb et al [15] developed a mathematical model to present the dynamical behavior of COVID-19 infection by incorporating isolation class. In our work we developed the SEIR model by adding the isolation class, then we formulated a system of five compartments with nonlinear incidence rate to prevent unboundedness of contact rate. The saturation factor may arise as a result of epidemic control, crowding of infectives, response of susceptible to disease severity or intervention measures to protect susceptibles. This study will lead to the mathematical model formulation in which the interaction of the exposed population and infected population occurred to the susceptible population. The infected persons must be sent to isolated class in different rates.

We have 8 sections, the model and description of its data is proposed in section 2. In section 3, we discussed the boundedness and positivity of the solutions of the system. In section 4, we found basic reproduction number and the local stability of the disease-free equilibrium is discussed. In section 5, global stability of the disease-free equilibrium is analysed. In section 6, local stability, and existence of positive equilibrium point. In section 7, equilibrium for special case. Finally, discussion in section 8.

2. THE MODEL

Our model consists of five compartments which are $S(t)$ for Susceptible individuals, $E(t)$ for Exposed individuals, $I(t)$ for Infected individuals, $\Phi(t)$ for Isolated infected individuals and $R(t)$ for Recovered individuals, so that the total population is given by

$$\mathrm{N(t)} = S(\mathrm{t}) + E(\mathrm{t}) + \mathrm{I(t)} + \Phi(\mathrm{t}) + \mathrm{R(t)} \tag{1}$$

The nonlinear system of differential equations that describe the transmission techniques of an infectious disease in the presence of educated infected individuals is given by

$$\begin{aligned}
S'(t) &= \pi - \left(\frac{\beta I(t)}{1+\alpha_1 I} + \frac{\beta\eta\Phi(t)}{1+\alpha_2\Phi}\right)S(t) - \mu S(t)\\
E'(t) &= \left(\frac{\beta I(t)}{1+\alpha_1 I} + \frac{\beta\eta\Phi(t)}{1+\alpha_2\Phi}\right)S(t) - (k+\mu+\bar{\gamma})E(t)\\
I'(t) &= kE(t) - (\sigma + \gamma_1 + \mu + \delta_1)I(t)\\
\Phi'(t) &= \bar{\gamma}E(t) + \sigma I(t) - (\mu + \gamma_2 + \delta_2)\Phi(t)\\
R'(t) &= \gamma_1 I(t) + \gamma_2\Phi(t) - \mu R(t)
\end{aligned} \tag{2}$$

where, π is the recruitment rate, β is the effective contact rate, $0 < \eta < 1$ is a parameter that accounts reduction in disease transmission by educated infected individuals in the I classs, k is the development rate of disease, γ_1, γ_2 are the natural recovery rates of the classes $I(t)$ and $\Phi(t)$ respectively, education rates of classes $I(t)$ and $E(t)$ are σ and $\bar{\gamma}$ respectively, δ_1 and δ_2 are death rate due to disease of the classes $I(t)$ and $\Phi(t)$ respectively such that $\delta_2 < \delta_1$, and the natural death is the rate μ. The model (2) extends some SEIR models in [16, 17, 18].

3. Boundedness and positivity of the solutions of the system

All the parameters of the system (2) are nonnegative because the system monitors human populations, and we will prove that in the following theorem.

Theorem 1. The parameters of the system (2) are nonnegative for all t > 0. (i.e. the solutions of the system with positive initial data will remain positive for t > 0).

Proof. Let $t_1 = sup\{t < 0 : S > 0, \quad E > 0, \quad I > 0, \Phi > 0, R > 0 \in [0, \quad t]\}$. So, $t_1 > 0$. From 1^{st} equation of (2)

Let $\lambda(t) = \frac{\beta I(t)}{1+\alpha_1 I} + \frac{\beta\eta\Phi(t)}{1+\alpha_2\Phi}$

$$\frac{dS}{dt} = \pi - \lambda(t)S(t) - \mu S(t) = \pi - (\lambda(t) - \mu)S(t) \tag{3}$$

Which can be rewritten as:

$$\frac{d}{dt}\left[S(t)e^{\mu t+\int_0^t \lambda(\tau)d\tau}\right] = \pi e^{\mu t+\int_0^t \lambda(\tau)d\tau} \tag{4}$$

Hence, by integration from $0 \to t_1$

$$S(t_1)e^{\mu t_1+\int_0^{t_1} \lambda(\tau)d\tau} - S(0) = \int_0^{t_1} \pi e^{\mu s+\int_0^s \lambda(\tau)d\tau} ds \tag{5}$$

Since $S(t) \geq S(t_1)$, so that

$$S(t) \geq S(0)e^{-\mu t_1-\int_0^{t_1} \lambda(\tau)d\tau}\left[S(0) + \int_0^{t_1} \pi e^{\mu s+\int_0^s \lambda(\tau)d\tau} ds\right] > 0 . \tag{6}$$

By the same way, it is easy to prove that $E, \quad I, \Phi$ and R are positive for all time $t > 0$.

We deduced the following result

Theorem 2. The following closed set is positively invariant

$$\mathcal{D} = \left\{(S, E, I, \Phi, R) \in R_+^5 : S + E + I + \Phi + R \leq \frac{\pi}{\mu}\right\}$$

Proof. By adding the five equations of the system (2),

$$N'(t) = \pi - \mu N - (\delta_1 I + \delta_2 \Phi)$$

Since $N'(t) \leq \pi - \mu N$, then $N'(t) \leq 0$ if $N(t) \geq \frac{\pi}{\mu}$. By the standard comparison theorem [19] it is obvious that

$$N(t) \leq N(0)e^{-\mu t} + \frac{\pi}{\mu}\left(1 - e^{-\mu t}\right),$$

in particular,

$$N(t) \leq \frac{\pi}{\mu} \quad \text{if} \quad N(0) \leq \frac{\pi}{\mu}.$$

Hence the region $\mathcal{D}$ is positively invariant.

Also, if $N(0) > \frac{\pi}{\mu}$, so either the solution enters $\mathcal{D}$ infinite time or $N(t)$ tends to $\frac{\pi}{\mu}$ asymptotically. therefore, $\mathcal{D}$ attracts all solutions in R_+^5.

Note. It is sufficient to consider the dynamics of the flow generated by the system (2) in $\mathcal{D}$ because the region $\mathcal{D}$ is positively invariant, where the usual existence, uniqueness, continuation result hold for the system [20].

4. Basic Reproduction Number and the local stability of the disease-free equilibrium

The disease-free equilibrium (D.F.E) of the system (2) is

$$\Gamma_0 = (S_0, E_0, I_0, \Phi_0, R_0) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0\right), \tag{7}$$

for the local stability of Γ_0 we will find the reproduction number using next generation method [21, 22]. Then

$$F = \begin{pmatrix} 0 & \frac{\beta\pi}{\mu} & \frac{\eta\beta\pi}{\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

$$V = \begin{pmatrix} \mu + k + \bar{\gamma} & 0 & 0 \\ -k & \mu + \sigma + \gamma_1 + \delta_1 & 0 \\ -\bar{\gamma} & -\sigma & \mu + \gamma_2 + \delta_2 \end{pmatrix} \tag{9}$$

the control reproduction number [22, 23], $R_0 = \rho(FV^{-1})$, (ρ is the spectral radius), is given

Let:

$$\left.\begin{aligned} K_1 &= \mu + k + \bar{\gamma} \\ K_2 &= \mu + \sigma + \gamma_1 + \delta_1 \\ K_3 &= \mu + \gamma_2 + \delta_2 \end{aligned}\right] \tag{10}$$

Using in [21, Theorem 2]. From (8) and (9), we have

$$V^{-1} = \begin{pmatrix} \frac{1}{K_1} & 0 & 0 \\ \frac{k}{K_1 K_2} & \frac{1}{K_2} & 0 \\ \frac{k\sigma + \bar{\gamma}K_2}{K_1 K_2 K_3} & \frac{-\sigma}{K_2 K_3} & \frac{k}{K_1 K_3} \end{pmatrix}$$

Then,

$$R_0 = \rho(FV^{-1}) = \frac{\pi\beta[kK_3 + \eta(k\sigma + \bar{\gamma}K_2)]}{\mu K_1 K_2 K_3}. \tag{11}$$

Theorem 2 in [22] used five hypothesis(A1)-(A5) and concluded that if $R_0 = \rho(FV^{-1})$ then the disease free equilibrium is locally asymptotically stable if $R_0 < 1$, but unstable if $R_0 > 1$. By using this theorem the following lemma is proved.

Lemma. The virus can disappear from the community when $R_0 < 1$, it means that the virus-free equilibrium of the system (2) with (7) is locally asymptotically stable and unstable if $R_0 > 1$.

5. Global stability of the disease-free equilibrium

To prove that the disease disappearance is independent of the initial sizes of sub-populations, we formulated the next theorem.

Theorem 3. The system (2) with (7), is global asymptotic stable in $\mathcal{D}$ if $R_0 \leq 1$.

Proof. Let next Lyapunov function:

$$\mathcal{F} = \left(\frac{kK_3 + \eta k\sigma + \eta\bar{\gamma}K_2}{\eta K_1 K_2}\right)E + \left(\frac{K_3 + \eta\sigma + \eta\bar{\gamma}K_2}{\eta K_2 k}\right)I + \Phi$$

Lyapunov derivative is given by

$$\mathcal{F}' = \left(\frac{kK_3 + \eta k\sigma + \eta\bar{\gamma}K_2}{\eta K_1 K_2}\right)E' + \left(\frac{K_3 + \eta\sigma + \eta\bar{\gamma}K_2}{\eta K_2 k}\right)I' + \Phi'$$

$$= \left(\frac{kK_3 + \eta k\sigma + \eta\bar{\gamma}K_2}{\eta K_1 K_2}\right)\left[\lambda S - \beta S\left(\frac{I}{1+\alpha_1 I} + \frac{\eta\Phi}{1+\alpha_2\Phi}\right) - K_1 E\right] + \left(\frac{K_3 + \eta\sigma + \eta\bar{\gamma}K_2}{\eta K_2 k}\right)(kE -$$

$$\leq \frac{kK_3 + \eta k\sigma + \eta\bar{\gamma}K_2}{\eta K_1 K_2}\left[\frac{\beta\pi}{\mu}(I + \eta\Phi) - K_1 E\right] + \frac{K_3 + \eta\sigma + \eta\bar{\gamma}K_2}{\eta K_2 k}(kE - K_2 I) + \sigma I - K_3\Phi$$

$$\leq \frac{\beta\pi(kK_3 + \eta k\sigma + \eta\bar{\gamma}K_2)}{\mu\eta K_1 K_2}(I + \eta\Phi) + \left(\sigma - \frac{K_3 + \eta\sigma + \eta\bar{\gamma}K_2}{\eta}\right)I - K_3\Phi$$

$$= \frac{K_3}{\mu(R_0 - 1)(I + \eta\Phi)}.$$

From theorem 1 we shew that all the parameters and variables of the system (2) are nonnegative, then $\mathcal{F}' \leq 0$ for $R_0 \leq 1$ with $\mathcal{F}' = 0$ iff $E = I = \Phi = 0$. Hence, $\mathcal{F}$ is a Lyapunov function in $\mathcal{D}$. Therefore, the largest compact invariant subset of the set where $\mathcal{F}' = 0$ is the singleton $\{(E, I, \Phi) = (0, 0, 0)\}$. Then, by LaSalle's invariance principle we have

$$(E, I, \Phi) \to (0, 0, 0) \text{ as } t \to \infty \tag{12}$$

Since $\lim_{t\to\infty} \sup I = 0$ and $\lim_{t\to\infty} \sup \Phi = 0$, then, for $\varepsilon > 0$ (sufficiently small), there exist constants $M_1 > 0$ and $M_2 > 0$ such that $\lim_{t\to\infty} \sup I \leq \varepsilon \ \forall t > M_1$ and $\lim_{t\to\infty} \sup \Phi \leq \varepsilon$ $\forall t > M_2$. Hence, from the last equation of the model (2), for $t > max\{M_1, M_2\}$,

$$R' \leq \gamma_1\varepsilon + \gamma_2\varepsilon - \mu R \tag{13}$$

Then, by comparison theorem [19]

$$R^\infty = \lim_{t\to\infty} \sup R \leq \frac{\gamma_1\varepsilon + \gamma_2\varepsilon}{\mu}, \tag{14}$$

let $\varepsilon \to 0$

$$R^\infty = \lim_{t\to\infty} \sup R \leq 0. \tag{15}$$

In the same way (using $\lim_{t\to\infty} \inf I = 0$ *and* $\lim_{t\to\infty} \inf \Phi = 0$). It is clear that

$$R_\infty = \lim_{t\to\infty} \inf R \geq 0, \tag{16}$$

then, it follows from (15), (16) that

$$R_\infty \geq 0 \geq R^\infty \tag{17}$$

so

$$\lim_{t\to\infty} R = 0 \tag{18}$$

Similarly,

$$\lim_{t\to\infty} S(t) = \frac{\pi}{\mu}, \tag{19}$$

from (13), (18) and (19), every solution of the equations of the system (2) with initial conditions in $\mathcal{D}$ approaches ε as $t \to \infty$ (for $R_0 < 1$).

6. Local stability and existence of positive equilibrium point

Since the first four equations of the system (2) are independent of R(t), by omitting the last equation for $R(t)$ without generally, the system (2) becomes

$$\begin{aligned} S'(t) &= \pi - \left(\frac{\beta I(t)}{1+\alpha_1 I} + \frac{\beta\eta\Phi(t)}{1+\alpha_2\Phi}\right)S(t) - \mu S(t) \\ E'(t) &= \left(\frac{\beta I(t)}{1+\alpha_1 I} + \frac{\beta\eta\Phi(t)}{1+\alpha_2\Phi}\right)S(t) - (k+\mu+\bar{\gamma})E(t) \\ I'(t) &= kE(t) - (\sigma + \gamma_1 + \mu + \delta_1)I(t) \\ \Phi'(t) &= \bar{\gamma}E(t) + \sigma I(t) - (\mu + \gamma_2 + \delta_2)\Phi(t) \end{aligned} \tag{20}$$

All solutions of (20) (S, E, I, Φ) nonnegative for $t \geq 0$.

We notice that the basic reproduction number R_0 controls the stability of system (20) and existence of a unique positive equilibrium on free equilibrium point say C_0. The free coronavirus equilibrium point in $C_0 = \left(\frac{\pi}{\mu}, 0, 0, 0\right)$.

Theorem 4. The system (20) is locally stable related to virus-free equilibrium point C_0 if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. For the local stability at C_0, the jacobian of system (20) is,

$$J = (E_0) = \begin{bmatrix} -\mu & 0 & -S\beta & -\eta S\beta \\ 0 & -(k+\mu+\bar{\gamma}) & 0 & 0 \\ 0 & k & -(\sigma+\gamma_1+\mu+\delta_1) & 0 \\ 0 & \bar{\gamma} & \sigma & -(\mu+\gamma_2+\delta_2) \end{bmatrix}$$

Which follows that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are negative, so the system (20) is locally stable for $R_0 < 1$ and unstable if $R_0 > 1$.

Theorem 5. There exists a unique virus equilibrium point $I^*(S^*, I^*, E^*, \Phi^*)$ for system (20), if $R_0 > 1$.

Proof. By letting the right hand side of all equations of system (20) to zero, as

$$\pi - \left(\frac{\beta I^*(t)}{1+\alpha_1 I^*} + \frac{\beta\eta\Phi^*(t)}{1+\alpha_2\Phi^*}\right)S^*(t) - \mu S^*(t) = 0 \tag{a}$$

$$\left(\frac{\beta I^*(t)}{1+\alpha_1 I^*} + \frac{\beta\eta\Phi^*(t)}{1+\alpha_2\Phi^*}\right)S^*(t) - (k+\mu+\bar{\gamma})E^*(t) = 0 \tag{b}$$

$$kE^*(t) - (\sigma + \gamma_1 + \mu + \delta_1)I^*(t) = 0 \tag{c}$$

$$\bar{\gamma}E^*(t) + \sigma I^*(t) - (\mu + \gamma_2 + \delta_2)\Phi^*(t) = 0 \tag{d}$$

From (c)

$$E^*(t) = \frac{\sigma + \gamma_1 + \mu + \delta_1}{k} I^*(t)$$

$$E^*(t) = \frac{K_2}{k} I^*(t) \text{ where } K_2 = \mu + \sigma + \gamma_1 + \delta_1$$

From (d)

$$\Phi^*(t) = \frac{\sigma I^*(t) + \bar{\gamma}\frac{K_2}{k} I^*(t)}{(\mu + \gamma_2 + \delta_2)} = \frac{\sigma k + \bar{\gamma} K_2}{k K_3} \text{ where } K_3 = \mu + \gamma_2 + \delta_2$$

From (a) and (b)

$$\pi - \mu S^*(t) - (k + \mu + \bar{\gamma}) E^*(t) = 0$$

$$S^*(t) = \frac{\pi}{\mu} - \frac{(k + \mu + \bar{\gamma})(\mu + \sigma + \gamma_1 + \delta_1)}{k} I^*(t)$$

$$S^*(t) = \frac{\pi}{\mu} - \frac{K_1 K_2}{k} I^*(t) \text{ where } K_1 = \mu + k + \bar{\gamma}$$

$$R_0 = \frac{\beta\pi}{\mu K_1 K_2 K_3} (k K_3 + \eta k \sigma + \eta \bar{\gamma} K_2)$$

I^* positives for all values I^*, E^*, Φ^*, S^* are positive if $\frac{\pi}{\mu} > \frac{K_1 K_2}{k} I^*$

It is obvious that all the values of S^*, E^*, Φ^* are positive for the condition

$$0 < I^* < \frac{\pi k}{\mu K_1 K_2}$$

or

$$I^* < \frac{\pi k}{\mu K_1 K_2} < \frac{R_0}{\beta}$$

if

$$1 < \beta I^* < \frac{\beta \pi k}{\mu K_1 K_2} < R_0$$

or

$$1 < \beta I^* < R_0.$$

Then S^*, E^*, Φ^* are positive.

7. Equilibrium for special case

The special case here when $\eta = 0$ which means if educated individuals do not transmit infection, in other words no isolated infections. To discuss the global asymptotically stability of the endemic equilibrium of the system (2) for this case, put $\eta = 0$ in system (2), we have

$$\begin{aligned} S'(t) &= \pi - \left(\frac{\beta I(t)}{1 + \alpha I} + \mu\right) S(t) \\ E'(t) &= \frac{\beta I(t)}{1 + \alpha I} S(t) - (k + \mu + \bar{\gamma}) E(t) \\ I'(t) &= k E(t) - (\sigma + \gamma_1 + \mu + \delta_1) I(t) \\ \Phi'(t) &= \bar{\gamma} E(t) + \sigma I(t) - (\mu + \gamma_2 + \delta_2) \Phi(t) \\ R'(t) &= \gamma_1 I(t) + \gamma_2 \Phi(t) - \mu R(t) \end{aligned} \tag{21}$$

It is easy to get the reproduction number of the system (21) as

$$\overline{R_0} = (R_0 | \eta = 0) = \frac{\beta \pi k}{\mu K_1 K_2}$$

we deduced the following theorem,

Theorem 6. The endemic equilibrium of the system (21) is unique and is globally asymptotic stable in $\mathcal{D}$ if $\overline{R_0} > 1$.

Proof. For the model (21), let $\overline{R_0} > 1$, then the associated equilibrium exists. Let a nonlinear Lyapunov function in the form:

$$\mathcal{F} = S - \bar{S} - \bar{S}ln\left(\frac{S}{\bar{S}}\right) + E - \bar{E} - \bar{E}\ln\left(\frac{E}{\bar{E}}\right) + \frac{K_1}{k}\left[I - \bar{I} - \bar{I}ln\left(\frac{I}{\bar{I}}\right)\right]$$

with Lyapunov derivative

$$\mathcal{F}' = S' - \frac{\bar{S}}{S}S' + E' - \frac{\bar{E}}{E}E' + \frac{K_1}{k}\left(I' - \frac{\bar{I}}{I}I'\right)$$

$$= \pi - \frac{\beta IS}{1+\alpha I} - \mu S - \frac{\bar{S}}{S}\left(\pi - \frac{\beta IS}{1+\alpha I} - \mu S\right) + \frac{\beta SI}{1+\alpha I} - K_1E - \frac{\bar{E}}{E}\left(\frac{\beta SI}{1+\alpha I} - K_1E\right)$$

$$+ \frac{K_1}{k}\left[kE - K_2I - \frac{\bar{I}}{I}(kE - K_2I)\right]$$

$$= \pi\left(1 - \frac{\bar{S}}{S}\right) - \mu S\left(1 - \frac{\bar{S}}{S}\right) + \frac{\beta SI}{1+\alpha I} - \frac{K_1K_2I}{k} - \frac{E\beta S}{E}\frac{I}{1+\alpha I} + K_1\bar{E} - K_1\frac{IE}{I} + \frac{K_1K_2I}{k}$$

From (21), at steady state,

$$\Pi = \left(\mu + \beta\frac{I}{1+\alpha I}\right)\bar{S}$$
$$k_1 = \frac{\beta}{E}\frac{I}{1+\alpha I}\bar{S} \qquad (22)$$
$$k_2 = \frac{kE}{I}$$

From (21) and (22)

$$\mathcal{F}' = \mu\bar{S}\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) - \frac{\bar{S}^2\frac{\bar{I}}{1+\alpha\bar{I}}}{S} + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}} + \beta\bar{S}\frac{I}{1+\alpha I} - \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}\frac{I}{\bar{I}} - \frac{\bar{E}\beta S}{E}\frac{I}{1+\alpha I}$$
$$+ \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}} - \frac{\beta\bar{S}\bar{I}^2}{I\bar{E}}\frac{E}{1+\alpha\bar{I}} + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}$$

Adding and subtracting $\beta\bar{S}\frac{I}{1+\alpha I}$ and $\frac{\beta S\bar{I}^2 I/\frac{\Pi}{1+\alpha I}}{(1+\alpha I)^2}$

$$\mathcal{F}' = \mu\bar{S}\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}\left(4 - \frac{\bar{S}}{S} - \frac{\bar{E}\frac{I}{1+\alpha I}}{E\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}} - \frac{\bar{I}E}{I\bar{E}} - \frac{\frac{\bar{I}I}{1+\alpha\bar{I}}}{\frac{I\bar{I}}{1+\alpha I}}\right) + \beta\bar{S}\frac{I}{1+\alpha I}$$
$$- \frac{\beta\bar{S}\frac{I\bar{I}}{1+\alpha\bar{I}}}{\bar{I}} - \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}} + \beta\bar{S}\frac{\bar{I}^2I}{(1+\alpha\bar{I})^2}$$

Gives

$$\mathcal{F}' = \mu\bar{S}\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}\left(4 - \frac{\bar{S}}{S} - \frac{\bar{E}S\dfrac{I}{1+\alpha I}}{E\bar{S}\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{\bar{I}E}{I\bar{E}} - \frac{\dfrac{\bar{I}I}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}\right)$$
$$+ \frac{(\pi-\mu\bar{S})\dfrac{I}{1+\alpha I}}{\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{(\pi-\mu\bar{S})I}{\bar{I}} - (\pi-\mu\bar{S}) + \frac{(\pi-\mu\bar{S})\dfrac{I\bar{I}}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}$$
$$= \mu\bar{S}\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}\left(4 - \frac{\bar{S}}{S} - \frac{\bar{E}\dfrac{I}{1+\alpha I}}{E\bar{S}\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{\bar{I}E}{I\bar{E}} - \frac{\dfrac{\bar{I}I}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}\right)$$
$$+ (\pi-\mu\bar{S})\left(\frac{\dfrac{I}{1+\alpha I}}{\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{I}{\bar{I}} - 1 + \frac{\dfrac{\bar{I}I}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}\right)$$

$$\mathcal{F}' = \mu\bar{S}\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) + \beta\bar{S}\frac{\bar{I}}{1+\alpha\bar{I}}\left(4 - \frac{\bar{S}}{S} - \frac{\bar{E}S\dfrac{I}{1+\alpha I}}{E\bar{S}\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{\bar{I}E}{I\bar{E}} - \frac{\dfrac{\bar{I}I}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}\right)$$
$$+ (\pi-\mu\bar{S})\frac{I}{\dfrac{I}{1+\alpha I}}\left(\frac{\dfrac{I}{1+\alpha I}}{I} - \frac{\dfrac{\bar{I}}{1+\alpha\bar{I}}}{\bar{I}}\right)\left(\frac{\dfrac{I}{1+\alpha I}}{\dfrac{\bar{I}}{1+\alpha\bar{I}}} - 1\right) \leq 0$$

since the arithmetic mean exceeds the geometric mean. Then,

$$\left(2 - \frac{\bar{S}}{S} - \frac{S}{\bar{S}}\right) \leq 0$$

$$\left(4 - \frac{\bar{S}}{S} - \frac{\bar{E}S\dfrac{I}{1+\alpha I}}{E\bar{S}\dfrac{\bar{I}}{1+\alpha\bar{I}}} - \frac{\bar{I}E}{I\bar{E}} - \frac{\dfrac{\bar{I}I}{1+\alpha\bar{I}}}{\dfrac{I\bar{I}}{1+\alpha I}}\right) \leq 0$$

then $\mathcal{F}' \leq 0$ for $\overline{R_0} > 1$ with $\mathcal{F}' = 0$ iff $S = \bar{S}, E = \bar{E}, I = I$, hence, $\mathcal{F}$ is a Lyapunov function on $\mathcal{D}$. Thus the largest compact invariant subset of the set where $\mathcal{F}' = 0$ is the singleton $\{(S, E, I) = (\bar{S}, \bar{E}, \bar{I})\}$, by the LaSalle's invariance principle [25, 26] that $S \to \bar{S}$, $E \to \bar{E}$, $I \to \bar{I}$ as $t \to \infty$.

Hence, $\limsup_{t\to\infty} \dfrac{I}{1+\alpha I} = \dfrac{\bar{I}}{1+\alpha\bar{I}}$, then, for sufficiently small $\varepsilon > 0$, there exist constant $T_1 > 0$ such that $\limsup_{t\to\infty} \dfrac{I}{1+\alpha I} \leq \dfrac{\bar{I}}{1+\alpha\bar{I}} + \varepsilon$ for all $t \in T_1$,

then, from the fourth equation of the system (21)

$$\Phi'(t) \leq \overline{\gamma E} + \sigma\left(\bar{I} + \varepsilon\right) - K_3\Phi$$

or

$$\Phi'(t) \leq \bar{\gamma}\frac{k_2}{k}\bar{I} + \sigma\left(\bar{I} + \varepsilon\right) - K_3\Phi$$

$$\Phi'(t) \leq \left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I} + \sigma\varepsilon - K_3\Phi$$

by comparison theorem [19].

$$\Phi^{\infty} = \lim_{t\to\infty} sup\ \Phi \leq \frac{\left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I} + \sigma\varepsilon}{K_3}$$

as $\varepsilon \to 0$, we have

$$\Phi^{\infty} \leq \frac{\left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I}}{K_3}, \tag{23}$$

similarly, by using

$$\lim_{t\to\infty} inf\ \Phi = \bar{\Phi}, \tag{24}$$

hence,

$$\Phi_{\infty} = \lim_{t\to\infty} inf\ \Phi \geq \frac{\left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I}}{K_3}$$

Thus. It follows from (23) and (24) that

$$\Phi_{\infty} \geq \frac{\left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I}}{K_3} \geq \Phi^{\infty}$$

Hence,

$$\lim_{t\to\infty}\Phi = \frac{\left(\frac{\bar{\gamma}k_2}{k} + \sigma\right)\bar{I}}{K_3} = \bar{\Phi}^{\infty}$$

Similarly, $\lim_{t\to\infty} R_0 = \overline{R_0}$, it follows that, every solution of each equation of system (21), with initial condition (1) approaches the equilibrium of the system as $t^{\to\infty}$ for $R_0 > 1$. Then the endemic equilibrium is unique.

8. DISSCUSSION.

From our discussion, we deduced that the close contact between the individuals is the main reason of spread of COVID-19. So, to reduce this risk we have to isolate the infected individuals. Here in our work, we have five classes: $S(t)$ for susceptible individuals, $E(t)$ for exposed individuals, $I(t)$ for Infected individuals, $\Phi(t)$ for Isolated individuals, and $R(t)$ for Recovered individuals. We formulated the model such that the exposed individuals interact with the exposed ones to the susceptible class. Also, we supposed that the infected individuals will be sent in different rates to isolated class.

We proposed an $SEI\Phi R$ epidemic model with isolation and nonlinear incidence rate. We deduced that the psychological or inhibition affect the behavioral changes of the susceptible persons when there is an increase in the number of infectious persons.

The reproduction figure of which (R_0) is based on a nonlinear incidence rate and adjusted for individual prophylactic activities. We investigate a formula to determine the reproduction number R_0. We deduced the points of the virus-free equilibrium and disease pandemic equilibrium. Then we studied the stability behaviour of each of them. We formulated our results in six theorems. In **Theorem 1** and **Theorem 2,** we found the boundedness and positivity of the solution of the system. We proved that the solutions of the system with positive initial data remain positive for $t > 0$, also the closed set $\mathcal{D}$ is positively invariant. In **Theorem 3**, we proved that the disease-free equilibrium is global asymptotic stable in $\mathcal{D}$ if $R_0 \leq 1$. In **Theorem 4,** we proved that the model is locally stable with respect to virus-free equilibrium point C_0 if $R_0 < 1$ and unstable if $R_0 > 1$. **Theorem 5,** we proved that there exists a unique virus equilibrium point for the model if $R_0 > 1$. Finally in **Theorem 6,** we proposed the equilibrium point for a special case when $\eta = 0$ for the system and we proved that the endemic equilibrium point for the modified model is unique and is globally asymptotic stable in $\mathcal{D}$ if $\overline{R_0} > 1$.

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نموذج وباء فيروس كورونا مع العزل ومعدل الحدوث اللاخطى

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الملخص:

لقد اقترحنا فى بحثنا نموذجًا وبائيًا يتكون من خمس فئات تصف مرض فيروس كورونا مع العزل. ولقد قمنا بتطوير نموذج SEIR لتقديم السلوك الديناميكى للعدوى من خلال دمج فئة العزل $\Phi(t)$ (لأن عزل الأفراد المصابين سيقلل من انتشار المرض). ناقشنا اولاً إيجابية النموذج وحدوده. ثم قمنا بحساب رقم التكاثر الأساسى للنموذج في ظل ظروف معينة. وقد لاحظنا أنه من خلال تحليل الاستقرار المحلى والعالمى لنموذجنا، فإن هذا الاستقرار يعتمد على رقم التكاثر الأساسى. ووجدنا أن النظام ليس لديه حالة توازن مستوطنة إذا كانت $R_0 < 1$. وأيضا ناقشنا الاستقرار العالمى باستخدام دالة ليبانوف من نوع جوه-فولتيرا مع مبدأ لاسيل للثبات وقد تبين أن النظام لديه نقطة توازن فريدة مستقرة عالمية مقاربة.