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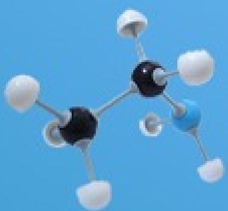

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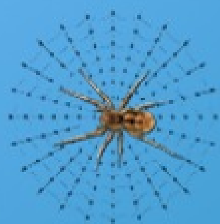

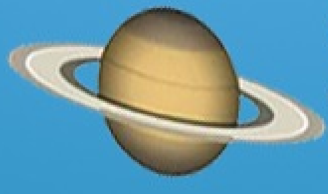

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ESTIMATING AND PREDICTION ACCELERATED LIFE TEST USING CONSTANT STRESS FOR MARSHALL-OLKINEXTENDED BURR TYPE X DISTRIBUTION BASED ON TYPE-II CENSORING

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ABSTRACT

In this paper constant stress accelerated life tests are discussed based on Type II censored sampling from Marshall-Olkin extended Burr Type X Distribution. The model parameters and the acceleration factor are estimated using the maximum likelihood estimation method and two-sample predictions are considered for future order statistics. Further, the asymptotic confidence intervals for the model parameters are discussed. Numerical study is given, and some interesting comparisons are presented to illustrate the theoretical results. Moreover, the results are applied on real dataset.

Keywords: Marshall-Olkin extended Burr Type X Distribution; Type II censored samples; two-sample prediction; maximum likelihood estimation and prediction.

1. Introduction

Rapid developments, improvements of the high technology, consumer's demands for highly reliable products and competitive markets have placed pressure on manufacturers to deliver products with high quality and reliability. In life testing, it is very difficult to estimate the time of failure for modern high reliability products such as electronics, power cables, metal fatigue, insulating materials, laser, airplane parts, aerospace vehicles, etc.; since these types of products are not likely to fail under usual operating conditions in the relatively short time available for test. For this reason, accelerated life testing (ALT) is preferred to be used in manufacturing industries to obtain enough failure data in a short period of time and necessary to study its relationship with external stress variables. Such testing could save much time, man power, material sources and money. In ALT the main assumption is that a life-stress relationship is known or can be assumed so that the data obtained from accelerated conditions can be extrapolated to usual conditions.

The stress can be applied in different ways like constant stress, step stress and progressive stress among others. In constant stress where under one higher than normal stress level, each specimen is run at a constant stress level. In practical use, most products run at constant stress as a constant stress test actual use, it is simple and has a lot of advantages over time-dependent stress loadings because most of real products are operated at a constant-stress condition. A constant stress ALT with different types of data and test planning has been studied by many authors, for example, Ismail [1], Hassan and AL-Thobety [2], AL-Dayian et al. [3] and Basak and Balakrishnan [4], among others.

Burr [5] introduced twelve different forms of cumulative distribution function for modeling lifetime data. Several authors considered different aspects of the Burr Type X distribution for example, Ahmad et al. [6], Surles and Padgett [7] derived the scaled Burr Type X distribution to fit the strength data, which was a comparative model to the Weibull distribution. Moreover, Raqab and Kundo [8] developed a two-parameter Burr Type X distribution that has a closed form of the generalized Rayleigh distribution. This distribution was further compared with the Weibull exponentiated exponential, gamma, and generalized exponential distributions. Aludaat et al. [9] used the Bayesian and non-Bayesian methods to estimate the parameters of Burr Type X distribution for Group Data.

Al-Saiari et al. [10] presented mathematical and statistical properties and limitations of Marshall-Olkin extended Burr Type X (MOEBX) distribution along with application to real lifetime data and provided graphical illustrations of the dimensions of MOEBX distribution. Also, they estimated the parameters using maximum likelihood (ML)and Bayesian methods.

The MOEBX distribution has not been applied based on Type II censoring and constant stress ALT in all the previous literature. The objective of this paper is based on studying the constant stress ALT for Type II censored sample data from MOEBX distribution. Further, the asymptotic confidence intervals (CIs) will be discussed in this paper.

This paper is organized as follows: Section 2 presents a brief summary about the MOEBX distribution and descriptions of the model. The ML and prediction for the MOEBX using constant stress life testing based on Type II censoring is obtained in Section 3. Numerical study is presented in Section 4. Finally, some general conclusions are introduced in Section 5.

2. The Marshall-Olkin Extended Burr Type X Distribution and Model Description

This section is divided into two subsections, Subsection 2.1 presents a brief summary about the MOEBX distribution and the description of the model and basic assumptions are presented in Subsection 2.2.

2.1 The Marshall-Olkin extended Burr Type X Distribution

Al-Saiariet al. [10] constructed a distribution with two shape parameters andpresented statistical properties of MOEBX distribution.

Assume that a random variable T has the MOEBX distribution and is denoted by
T ~ MOEBX (α, k). The cumulative distribution function (cdf) and the probability density function (pdf) for MOEBX distribution are, respectively, written as:

$$F(t;\alpha,k)=\frac{\left(1-e^{-t^2}\right)^k}{\left[1-(1-\alpha)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]},\qquad t>0,\alpha,k>0, \tag{1}$$

and

$$f(t;\alpha,k)=\frac{2\alpha k t\, e^{-t^2}\left(1-e^{-t^2}\right)^{k-1}}{\left[1-(1-\alpha)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]^2},\qquad t>0,\ \alpha,k>0, \tag{2}$$

where $\alpha,k>0$are shape parameters.

The reliability function (rf) and hazard rate function (hrf) of the MOEBX distribution are, respectively, given by

$$R(t;\alpha,k)=1-\frac{\left(1-e^{-t^2}\right)^k}{\left[1-(1-\alpha)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]},\qquad t>0,\ \alpha,k>0, \tag{3}$$

and

$$h(t;\alpha,k) = \frac{2kte^{-t^2}\left(1-e^{-t^2}\right)^{k-1}}{\left\{1-(1-\alpha)\left[1-\left(1-e^{-t^2}\right)^k\right]\right\}\left[1-\left(1-e^{-t^2}\right)^k\right]}, \qquad t>0,\ \alpha,k>0, \tag{4}$$

For more details about the properties of this distribution (see Al-Saiariet al. [10]).

2.2. Model Description

It is assumed that the stress V_j affects only the shape parameter $\boldsymbol{\alpha}$ of the MOEBX distribution to be$\boldsymbol{\alpha}_j$through a certain acceleration inverse power law model. This model takes the following form:

$$\alpha_j = cs_j^{-p}, \qquad c>0, p>0, \tag{5}$$

where

$$S_j = \frac{V^*}{V_j}, \qquad j=1,2, \quad V^* = \prod_{j=1}^{2} V_j^{b_j}, \quad b_j = \frac{r_j}{\sum_{j=1}^{2} r_j}, \tag{6}$$

c is the constant of proportionality, p is the power of the applied stress,$S_j$is the applied stress and b_j is the power of the stress level V_j

Suppose that the life testing experiment is carried out through two levels of high stresses $V_j, j=1,2$.Assuming that V_u is the usually used condition such that at each stress level $V_j, j=1,2$.There are $n_j$items put on test. When Type-II censoring is adopted at each stress level, the experiment terminates once the number of failures $r_j$out of items $n_j$are reached.

In this paper, the lifetimes of test items are assumed to have a MOEBX distribution. The cdf and pdf for an item tested at accelerated conditions are given by:

$$\boldsymbol{F}(\boldsymbol{t};\boldsymbol{c},p,k) = \frac{\left(1-e^{-t^2}\right)^k}{\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]}, \qquad t>0;\ \boldsymbol{c},p,k>0, \tag{7}$$

and

$$\boldsymbol{f}(\boldsymbol{t};\boldsymbol{c},p,k) = \frac{2cs_j^{-p}kxe^{-t^2}\left(1-e^{-t^2}\right)^{k-1}}{\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]^2}. \quad t>0;\ \boldsymbol{c},p,k>0, \tag{8}$$

The rf and hrf for an item tested at accelerated conditions are given as follows:

$$\boldsymbol{R}(t;\boldsymbol{c},p,k) = 1-\frac{\left(1-e^{-t^2}\right)^k}{\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]} \qquad t>0;\ \boldsymbol{c},p,k>0, \tag{9}$$

and

$$\boldsymbol{h}(t;\boldsymbol{c},p,k) = \frac{2cs_j^{-p}k\,t\,e^{-x^2}\left(1-e^{-t^2}\right)^{k-1}}{\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t^2}\right)^k\right]\right]\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t^2}\right)^k\right]-\left(1-e^{-t^2}\right)^k\right]},$$

$$t,c,p,k>0. \tag{10}$$

3. Maximum Likelihood Estimation and Predication

In this section, the estimation of the unknown parameters and rf for MOEBX distribution in constant stress ALT under Type II censored data are discussed in Subsection 3.1.ML point estimation and CIs are obtained in Subsection 3.2, 3.3, respectively. Two-sample prediction for MOEBX distribution in constant stress ALT under Type II censored data is considered in Subsection 3.4.

3.1 Maximum likelihood estimation

Suppose that the life testing experiment is carried out through two levels of high stresses $V_j, j = 1,2$ and the V_u is the usually used condition such that $V_u < V_1 < V_2$.The likelihood function of observations $t_{ij}, i = 1,2, \dots, r_j$ and $j = 1,2$ is given as:

$$L_j = \prod_{j=1}^{2} \frac{n_j!}{(n_j - r_j)!} \left[\prod_{i=1}^{r_j} f(t_{ij}) \left[1 - F\left(t_{r_j j}\right) \right]^{n_j - r_j} \right], \tag{11}$$

Substituting (7) and (8) in (11) then the likelihood function can be written as:

$$L(c, p, k; \underline{t}) = \prod_{j=1}^{2} \left[A_j \prod_{i=1}^{r_j} \frac{2 c s_j^{-p} k t_{ij} e^{-t_{ij}^2} \left(1 - e^{-t_{ij}^2}\right)^{k-1}}{\left[1 - \left(1 - c s_j^{-p}\right)\left[1 - \left(1 - e^{-t_{ij}^2}\right)^k\right]\right]^2} [1 - B_j]^{n_j - r_j} \right], \tag{12}$$

where

$$A_j = \frac{n_j!}{(n_j - r_j)!} \text{and } B_j = \frac{\left(1 - e^{-t_{r_j j}^2}\right)^k}{c s_j^{-p} \left(1 - \left(1 - e^{-t_{r_j j}^2}\right)^k\right) + \left(1 - e^{-t_{r_j j}^2}\right)^k}.$$

3.1.1 Point estimation

The ML estimators of c, p and k are obtained by maximizing the logarithm of the likelihood function, denoted by ℓ which can be written in the form

$$\begin{aligned} \ell \equiv lnL(c, p, k; \underline{t}) & \\ \propto & \ln(2) \sum_{j=1}^{2} r_j + \ln(c) \sum_{j=1}^{2} r_j \\ & - p \sum_{j=1}^{2} r_j \ln(s_j) + \ln(k) \sum_{j=1}^{2} r_j + \sum_{j=1}^{2} \sum_{i=1}^{r_j} \ln(t_{ij}) - \sum_{j=1}^{2} \sum_{i=1}^{r_j} t_{ij}^2 \\ & + (k-1) \sum_{j=1}^{2} \sum_{i=1}^{r_j} \ln\left(1 - e^{-t_{ij}^2}\right) \\ & - 2 \sum_{j=1}^{2} \sum_{i=1}^{r_j} \ln\left[c s_j^{-p} \left(1 - \left(1 - e^{-t_{ij}^2}\right)^k\right) + \left(1 - e^{-t_{ij}^2}\right)^k\right] \\ & + \sum_{j=1}^{2} (n_j - r_j)\, ln \left[1 - \frac{\left(1 - e^{-t_{r_j j}^2}\right)^k}{c s_j^{-p} \left(1 - \left(1 - e^{-t_{r_j j}^2}\right)^k\right) + \left(1 - e^{-t_{r_j j}^2}\right)^k}\right]. \end{aligned} \tag{13}$$

The partial first derivatives of the logarithm of the likelihood function with respect to c, p and k are given below:

$$\frac{\partial \ell}{\partial c}=\frac{1}{c}\sum_{j=1}^{2} r_j-2\sum_{j=1}^{2}\sum_{i=1}^{r_j}\frac{s_j^{-p}\left(1-\left(1-e^{-t_{ij}^{2}}\right)^{k}\right)}{\left[cs_j^{-p}\left(1-\left(1-e^{-t_{ij}^{2}}\right)^{k}\right)+\left(1-e^{-t_{ij}^{2}}\right)^{k}\right]}$$
$$-\sum_{j=1}^{2}\frac{\left(n_{j-r_j}\right)\left(1-e^{-t_{r_j j}^{2}}\right)^{k}s_j^{-p}\left(1-\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right)}{(1-B_j)\left[cs_j^{-p}\left(1-\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right)+\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right]^{2}}, \quad (14)$$

$$\frac{\partial \ell}{\partial p}=-\sum_{j=1}^{2} r_j(\ln(s_j))+2c\sum_{j=1}^{2}\sum_{i=1}^{r_j}\frac{s_j^{-p}(\ln(s_j))\left(1-\left(1-e^{-t_{ij}^{2}}\right)^{k}\right)}{\left[cs_j^{-p}\left(1-\left(1-e^{-t_{ij}^{2}}\right)^{k}\right)+\left(1-e^{-t_{ij}^{2}}\right)^{k}\right]}$$
$$-c\sum_{j=1}^{2}\frac{\left(n_{j-r_j}\right)\left(1-e^{-t_{r_j j}^{2}}\right)^{k}s_j^{-p}(\ln(s_j))\left(1-\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right)}{(1-B_j)\left[cs_j^{-p}\left(1-\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right)+\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right]^{2}}, \quad (15)$$

and

$$\frac{\partial \ell}{\partial k}=\frac{1}{k}\sum_{j=1}^{2} r_j+\sum_{j=1}^{2}\sum_{i=1}^{r_j} ln\left(1-e^{-t_{ij}^{2}}\right)-\frac{2E_{ij}}{H_j}-\frac{K_j}{(1-B_j)[D_j]^{2}} \quad (16)$$

where

$$\boldsymbol{N_{ij}}=\left(\boldsymbol{1}-\boldsymbol{e}^{-t_{ij}^{2}}\right),$$

$$M_{ij}=\left(cs_j^{-p}\left(1-\left(1-e^{-t_{ij}^{2}}\right)^{k}\right)+\left(1-e^{-t_{ij}^{2}}\right)^{k}\right),$$

$$\boldsymbol{E_{ij}}=\sum_{j=1}^{2}\sum_{i=1}^{r_j}\left(1-e^{-t_{ij}^{2}}\right)^{k}\ln\left(1-e^{-t_{ij}^{2}}\right)\left(1-cs_j^{-p}\right),$$

$$D_j=cs_j^{-p}\left(1-\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\right)+\left(1-e^{-t_{r_j j}^{2}}\right)^{k},$$

$$G_j=\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\left[cs_j^{-p}-1\right],$$

$$H_{j=}\left[\left(1-e^{-t_{ij}^{2}}\right)^{k}\left(1-cs_j^{-p}\right)+cs_j^{-p}\right],$$

and

$$K_j=\sum_{j=1}^{2}\left(n_{j-r_j}\right)\left(1-e^{-t_{r_j j}^{2}}\right)^{k}\ln\left(1-e^{-t_{r_j j}^{2}}\right)\left[D_j+G_j\right].$$

The ML estimators are obtained by setting (14)-(16) to zeros. The system of non-linear equations can be solved numerically using Newton-Raphson method, to obtain the ML estimates of $\hat{c}, \hat{p}$and $\hat{k}$.Depending on the invariance property of the ML estimators, then, the ML estimators of the rf and hrf are obtained by replacing the parameters c, p and$k$in (9) and (10) respectively by their ML estimators.

For a given value of$t$, the ML estimators of $R(t)$ and $h(t)$are given as follows:

$$\hat{R}(t) = 1 - \frac{\left(1-e^{-t^2}\right)^{\hat{k}}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-t^2}\right)^{\hat{k}}\right]\right]}, \qquad t, c, p, k > 0. \tag{17}$$

and

$$\hat{h}(t) = \frac{2\hat{c}s_j^{-\hat{p}}\hat{k}te^{-t^2}\left(1-e^{-t^2}\right)^{\hat{k}-1}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-t^2}\right)^{\hat{k}}\right]\right]\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-t^2}\right)^{\hat{k}}\right]-\left(1-e^{-t^2}\right)^{\hat{k}}\right]}, \qquad t, c, p, k > 0. \tag{18}$$

Practically, it is difficult to use the results obtained at accelerated conditions to make prediction about the product performance over time at the use or design conditions. Considering prediction from an ALT, one must put strong assumptions about the adequacy of the ALT process to describe the use process. Selection of the accelerated model is the most important difficulty. This model relates one or more parameters to the stress levels that are to be applied to the testing items. It should be physically suitable for the item or product being tested and the type of stress being applied to accelerate failures, see Padgett [11].
Depending on the invariance property of ML estimators, the value of the parameter $\alpha_u$under stress Vu and the ML estimator of the rf under usual conditions at a mission time $t_0$could be predicted. The ML estimator of the parameter of MOEBX distribution $\hat{\alpha}_u$ can be derived by using the following equation:

$$\hat{\alpha}_u = \hat{c}s_u^{-\hat{p}}. \tag{19}$$

where

$$S_u = \frac{V^*}{V_u}.$$

Also, the ML estimator of the rf under usual conditions is given by:

$$\hat{R}(t_0) = 1 - \frac{\left(1-e^{-t_0{}^2}\right)^{\hat{k}}}{\left[1-(1-\hat{\alpha}_u)\left[1-\left(1-e^{-t_0{}^2}\right)^{\hat{k}}\right]\right]}. \tag{20}$$

The parameter$\alpha_u$and the rf at different mission timest_0 are predicted under design stress $V_u < V_1$.

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3.1.2 Confidence intervals

The asymptotic variance covariance matrix (AVCM) of the estimators $\hat{c}, \hat{p}$ and $\hat{k}$ are obtained depending on the inverse asymptotic Fisher information matrix (AFIM) using the partial second derivatives of the logarithm of the likelihood function.

The AFIM can be written as follows:

$$\tilde{I} \approx -\left[\frac{\partial^2 l}{\partial\psi_i \partial\psi_j}\right], \text{ where} \qquad i,j = 1,2,3, \tag{21}$$

where $\psi_1 = \boldsymbol{c}, \psi_2 = p,$ and $\psi_3 = \boldsymbol{k}$.

For large sample size, the ML estimators under regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the asymptotic confidence interval (ACI) for the parameters; ψ, can be obtained by

$\mathrm{P}\left[-\mathrm{Z} < \frac{\hat{\psi}_{\mathrm{iML}} - \psi_{\mathrm{i}}}{\sigma_{\hat{\psi}_{\mathrm{iML}}}} < \mathrm{Z}\right] = 1 - \tau$, where Z is the $100\left(1 - \frac{\tau}{2}\right)$th standard normal percentile. The two-sided approximate $100(1 - \tau)\%$ the confidence intervals are

$$\mathrm{L}_{\psi_i} = \hat{\psi}_{\mathrm{iML}} - \mathrm{Z}_{\frac{\tau}{2}} \hat{\sigma}_{\hat{\psi}_{\mathrm{iML}}}, \qquad \text{and} \qquad \mathrm{U}_{\psi_i} = \hat{\psi}_{\mathrm{iML}} + \mathrm{Z}_{\frac{\tau}{2}} \hat{\sigma}_{\hat{\psi}_{\mathrm{iML}}}, \tag{22}$$

where $\hat{\sigma}_{\hat{\psi}_{\mathrm{iML}}}$ is the standard deviation and $\hat{\psi}_{\mathrm{iML}}$ *is* $\hat{\boldsymbol{c}}, \hat{p}, \hat{k}$ or $\hat{\boldsymbol{\alpha}}$, respectively.

3.2 Maximum likelihood prediction

In this subsection two-sample prediction is considered. ML point and interval prediction for a future observation from the constant stress accelerated life test (CSALT) based on Type II censoring for MOEBX distribution are discussed.

Assuming that $\mathrm{X}_{(1)} \leq \mathrm{X}_{(2)} \leq \cdots \leq \mathrm{X}_{(r)}$ are the first r ordered life times in a random sample of n components Type II censoring whose failure times are identically distributed as a random variable X having the pdffor an item tested at accelerated conditions is given by (8) which is an informative sample, and that $T_{(1)}, T_{(2)}, \dots, T_{(m)}$ is a future independent random sample (of size m) from the same distribution. Our aim is to predict a statistic in the future sample based on the informative sample [see Kaminsky and Rhodin [12], Ateya and Mohammed [13] and Raqab et al. [14]].

For the future sample of size m, let $T_{(s)}$ denotes the s^{th} order statistic, $1 \leq s \leq m$, then the pdf of $T_{(s)}$ is given by

$$f_{s:m}\left(t_{(s)}; \underline{\omega}\right) = C(s) f\left(t_{(s)}; \underline{\omega}\right)\left[F\left(t_{(s)}; \underline{\omega}\right)\right]^{s-1}\left[1 - F\left(t_{(s)}; \underline{\omega}\right)\right]^{m-s}, \quad t_{(s)} > 0, \tag{23}$$

$$\underline{\omega} = (c, p, k) \quad , C_{s:m} = \frac{1}{B(s, m-s+1)}, \mathrm{s} = 1,2,3, \dots, \mathrm{m},$$

Using the binomial expansion theorem for $\left[1 - F\left(t_{(s)}; \underline{\omega}\right)\right]^{m-s}$, yields

$$f_{s:m}\left(t_{(s)}; \underline{\omega}\right) = C_{s:m} f\left(t_{(s)}; \underline{\omega}\right) \sum_{j=0}^{m-s} (-1)^j \binom{m-s}{j} \left[F\left(t_{(s)}; \underline{\omega}\right)\right]^{s+j-1}, \tag{24}$$

and substituting (7) and (8) into (24), then one can obtain the pdf of s^{th} order statistic for an item tested at accelerated conditions:

$$f_{s:m}\left(t_{(s)};\underline{\omega}\right)$$
$$= 2cs_j^{-p}kt_{(s)}e^{-t_{(s)}^2}\sum_{j=0}^{m-s}C_{s:m}(-1)^j\binom{m-s}{j}\frac{\left(1-e^{-t_{(s)}^2}\right)^{k(s+j)-1}}{\left[1-\left(1-cs_j^{-p}\right)\left[1-\left(1-e^{-t_{(s)}^2}\right)^k\right]\right]^{s+j+1}}$$
$$t_{(s)} > 0, \quad (25)$$

Assuming that the parameters $\underline{\omega}$ are unknown and independent, and then the ML prediction density (MLPD) of $T_{(s)}$ given $\underline{\widehat{\omega}}_{ML}$ can be obtained using the conditional pdf of the s^{th} order statistic which is given by (25) after replacing the vector of parameters $\underline{\omega}$ by their ML estimators $\underline{\widehat{\omega}}_{ML}$, as follows:

$$f_{s:m}\left(t_{(s)};\underline{\widehat{\omega}}_{ML}\right) =$$
$$2\hat{c}s_j^{-\hat{p}}\hat{k}t_{(s)}e^{-(t_{(s)})^2}\sum_{j=0}^{m-s}C_{s:m}(-1)^j\binom{m-s}{j}\frac{\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}(s+j)-1}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}}\right]\right]^{s+j+1}}$$
$$, t_{(s)} > 0, \quad (26)$$

where

$$\underline{\widehat{\omega}}_{ML} = \left(\hat{c},\hat{p},\hat{k}\right) \;, \quad (27)$$

3.2.1. Point predication

The ML predictor (MLP) for the future observation $T_{(s)}$, based on Type II censoring can be derived using (26) as follows:

$$\hat{t}_{(s)(ML)} = E\left(t_{(s)};\underline{\widehat{\omega}}_{ML}\right) = \int_{t_{(s)}} t_{(s)}f_{s:m}\left(t_{(s)};\underline{\widehat{\omega}}_{ML}\right)dt_{(s)}$$
$$= 2\sum_{j=0}^{m-s}C_{s:m}(-1)^j\binom{m-s}{j}\int_{t_{(s)}}\hat{c}s_j^{-\hat{p}}\hat{k}(t_{(s)})^2e^{-(t_{(s)})^2}\frac{\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}(s+j)-1}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}}\right]\right]^{s+j+1}}dt_{(s)} \quad (28)$$

3.2.2 Interval prediction

A $100(1-\tau)\%$ ML predictive bound (MLPB) for the future observation $T_{(s)}$, such that $P\left(L_{(s)}(\underline{x}) < T_{(s)} < U_{(s)}(\underline{x})|\underline{x}\right) = 1-\tau$, are given by:

$$P(T_{(s)} > L_{(s)}(\underline{x})|\underline{x}) = \int_{L_{(s)}(\underline{x})}^{\infty}f_{s:m}\left(t_{(s)};\underline{\widehat{\omega}}_{ML}\right)dt_{(s)} = 1-\frac{\tau}{2}, \quad (29)$$

and

$$P(T_{(s)} > U_{(s)}(\underline{x})|\underline{x}) = \int_{U_{(s)}(\underline{x})}^{\infty}f_{s:m}\left(t_{(s)};\underline{\widehat{\omega}}_{ML}\right)dt_{(s)} = \frac{\tau}{2}. \quad (30)$$

Substituting (26) in (29) and (30), the MLPB are obtained as follows:

$$P(T_{(s)} > L_{(s)}(\underline{x})|\underline{x})$$

$$= 2\sum_{j=0}^{m-s} C_{s:m}(-1)^j \binom{m-s}{j} \int_{L_{(s)}(\underline{x})}^{\infty} \hat{c}s_j^{-\hat{p}} \hat{k} e^{-(t_{(s)})^2} \frac{\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}(s+j)-1}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}}\right]\right]^{s+j+1}} dt_{(s)}$$

$$= 1-\frac{\tau}{2}, \tag{31}$$

and

$$P(T_{(s)} > U_{(s)}(\underline{x})|\underline{x})$$

$$= 2\sum_{j=0}^{m-s} C_{s:m}(-1)^j \binom{m-s}{j} \int_{U_{(s)}(\underline{x})}^{\infty} \hat{c}s_j^{-\hat{p}} \hat{k} e^{-(t_{(s)})^2} \frac{\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}(s+j)-1}}{\left[1-\left(1-\hat{c}s_j^{-\hat{p}}\right)\left[1-\left(1-e^{-(t_{(s)})^2}\right)^{\hat{k}}\right]\right]^{s+j+1}} dt_{(s)}$$

$$= \frac{\tau}{2}. \tag{32}$$

Remark:

If $s = 1$, $s = m$ and $s = \frac{m+1}{2}$ in (28), one can predict the minimum observable, $T_{(1)}$, the maximum observable, $T_{(m)}$, and the median observable when m is odd, $T_{\left(\frac{m+1}{2}\right)}$.

4. Numerical Study

This section aims to investigate the precision of the theoretical results of estimation and prediction on the basis of simulated and real data.

4.1 Simulation algorithm

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates on the basis of generated data from the MOEBX distribution considering the CSALT. The ML averages of the parameters and rf based on Type II censoring are computed. Moreover, CIs of the parameters is calculated. Also, ML two-sample predictors point and interval are calculated. Simulation studies are performed using Mathcad15 for illustrating the obtained results.

The steps of the simulation procedure based on Type II censored data are as follows:

a. The transformation of several data sets generated from 2- parameter MOEBX distribution for a combination of the population parameter values between uniform distribution and MOEBX distribution is given as follows:
b. It is assumed that only two different levels of stress, (k = 2), V_1= 1 and V_2= 2.5, which are higher than the stress at usual condition, V_u= 0.5.
c. For each sample size n sort x_i`s, such that $x_1 \le x_2 \le \cdots \le x_{(n)}$.
d. Choose the number of failures r to be less than n(r$< n$).
e. The population parameter values of c, p, and k are used in this simulation study.
f. Repeat all the previous steps N times, where N represents a fixed number of simulated samples.

g. A computer program is used, for deriving the number of units r and the population parameter values, depending on Mathcad 15 and using a computer program is derived depending on Mathcad 15 using the iterative technique of Newton Raphson method to solve the derived nonlinear logarithmic likelihood (14)-(16) simultaneously.
h. Once the values of $\hat{c}, \hat{p}$and $\hat{k}$ are obtained and these estimates, under the stress are used to estimate the shape parameter, which is estimated as under this stress V_u= 0.5, is estimated as$\hat{\alpha}_u = \hat{c}S_u^{-\hat{p}}$.
i. Also, the rf is estimated for different values of mission times under usual condition.
j. The CIs of the parameters are obtained.
k. Evaluating the performance of the estimators of c, p, and k has been considered through some measurements of accuracy. In order to study the precision and variation of ML estimates, it is convenient to use

   The relative absolute bias (RAB) $= \frac{|\text{estimate-population parameter}|}{\text{population parameter}}$= and the estimated risk (ER)$= \frac{\sum_{i=0}^{N}(estimate - true\ value)^2}{N}$.
l. The ML predictor points and interval for a future observation from the MOEBX distribution based on Type II censored data are computed for the two-sample case.
m. Simulation results of the ML estimates are displayed in Tables 1 and 2, where the samples of size (n =30, 60, 100), are used. For each sample size, the censoring level is 20% and the chosen parameters value are selected as (Case 1, $c = 0.02, k = 0.236, p = 0.002,\ n_1 = 15,\ n_2 = 15, r_1 = 3\ and\ r_2 = 3$),(Case 2, $c = 0.02,\ k = 0.236, p = 0.002,\ n_1 = 30,\ n_2 = 30,\ r_1 = 6\ and\ r_2 = 6$) and (Case 3, $c = 0.02, k = 0.236, p = 0.002, n_1 = 50, n_2 = 50, r_1 = 10\ and\ r_2 = 10$).

   Table 1 displays the ML averages, ERs and CIs of the unknown parameters, based on Type II censoring in Case 1, Case 2 and Case 3, respectively. While Table 2 presents the asymptotic variance-covariance matrix for unknown parameters ofc, k and p based on Type II censoring.
n. Table 3 presents the predicted values of the parameter α_u under the usual condition stress V_u using (19). The rf is also predicted for different values of mission time using (20).
o. The ML two-sample predictors are presented in Table 4.

4.2 Application to real life data

The main aim of this subsection is to demonstrate how the proposed method can be used in practice. The real lifetime data set is used for this purpose. The data were first given in Lawless [15]. The data represent the breaking strength T of single carbon fibers of different lengths which consisted of 250 observations. The data, from Crowder [16], are used in Example 6.4.2 and give the breaking strengths of single carbon fibers of different lengths. Length (E) Breaking Strength (t)

Al-Saiari et al. [10] tested this data using Kolmogorov-Smirnov test and stated that it follows MOEBX distribution. The K-S test statistic is 0.1571 with the p-value = 0.3531, from p value showed that the proposed model fits the data very well.

- In this Section, the numerical illustration to obtain the MLE of the unknown parameters c, p, k and the estimates of the parameterα and the reliability function $R(t_0)$ under usual conditions V_u using Type-II.

- Table 5 gives the ML averages of the unknown parameters. While Table 6 presents the predicted value of the parameter α_u under the usual condition stress V_u is predicted for different values of c, k and p using (19). The reliability function is also predicted for different values of mission time using (20) for the real data set based on Type II censoring**.**
- Table 7 presents the ML two-sample predictors for the future observation based on Type II censoring of the real dataset.

Table 1: ML averages, estimated risks, relative absolute biases and 95% CIs for the parameters $c, p, \mathrm{k}, \boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ based on Type II censoring, (rep=300, r=20% in sample size, $\boldsymbol{V_1} = 1, \boldsymbol{V_2} = 2.5, \boldsymbol{c} = \mathbf{0.02}, \boldsymbol{p} = \mathbf{0.002}$ ***and*** $\boldsymbol{k} = \mathbf{0.236})$

N	n_1	n_2	r_1	r_2	Parameters	Averages	RAB	ERs	LL	UL	length
30	15	15	3	3	c	0.010	0.510	0.001	0.00	0.02	0.02
					k	0.120	0.480	0.020	0.080	0.170	0.090
					p	0.030	11.82	0.020	0.000	0.090	0.090
					α_1	0.009	0.540	0.005	0.018	0.009	0.001
					α_2	0.011	0.460	0.007	0.01	0.011	0.010
60	30	30	6	6	c	0.017	0.16	0.001	0.008	0.025	0.017
					k	0.091	0.62	0.032	0.054	0.127	0.073
					p	0.082	40.16	0.103	0.000	0.190	0.190
					α_1	0.015	0.250	0.001	0.014	0.016	0.002
					α_2	0.020	0.002	0.002	0.019	0.021	0.002
100	50	50	10	10	c	0.001	0.935	0.001	0.000	0.005	0.005
					k	0.087	0.633	0.01	0.065	0.108	0.043
					p	0.002	0.022	5E-8	0.002	0.0021	0.0001
					α_1	0.001	0.935	0.001	0.001	0.0013	0.0003
					α_2	0.001	0.935	0.001	0.001	0.0013	0.0003

Table 2: The asymptotic variance-covariance matrix for the parameters c, p **and** k based on Type II censoring, (rep=300, r=**2**0% in sample size, $\boldsymbol{V_1} = 1, \boldsymbol{V_2} = 2.5, \boldsymbol{c} = \mathbf{0.02}, \boldsymbol{p} = \mathbf{0.002}$ ***and*** $\boldsymbol{k} = \mathbf{0.236})$

N	n_1	n_2	r_1	r_2	Parameters	C	K	P
30	15	15	3	3	c	4.033E-2	-1.001E-1	1.406E-3
					k		9.042E-2	-2.084E-2
					P			0.020
60	30	30	6	6	c	5.085E-3	-4.044E-3	3.497 E-3
					k		0.011	-7.014 E-3
					P			0.096
100	50	50	10	10	c	1.723E-4	-9.304E-5	1.372 E-6
					k		6.195E-3	-7.449 E-7
					P			4.799 E-8

Table 3: Estimates of α_u and $R_u(t_0)$ under usual Conditions $V_1 = .5$ and $V_2 = .5$ for different time t_0 at different sample sizes based on Type II censoring

n	n_1	n_2	r_1	r_2	$\hat{\alpha}_u$	t_0	$\hat{R}_u(t_0)$
30	15	15	3	3	0.02	.01 .06 .11 .16	0.019 0.009 0.006 0.005
60	30	30	6	6	0.02	.01 .06 .11 .16	0.020 0.010 0.008 0.006
100	50	50	10	10	0.02	.01 .06 .11 .16	0.002 0.001 0.001 0.001

Table 4: ML predictive and bounds of the future observation based on Type II censoring under two-sample prediction (rep=300, r=20% in sample size,
$V_1 = 1, V_2 = 2.5, c = 0.01, p = 0.03$ *and* $k = 0.12$)

s	$\hat{t}_{(s)}(ML)$	LL	UL	length
1 15 29	0.00 6.13E-8 2.523E-3	0.00 0.00 5.994E-4	1.27E-11 1.33E-7 4.446E-3	1.27E-11 1.33E-7 0.0035
1 15 29	2.912 E-12 1.138 E-7 4.578 E-3	0.00000 5.187E-8 3.056E-3	1.891E-11 1.757E-7 6.101E-3	1.891E-11 1. E-6 0.0031
1 13 25	0.000 9.215 E-11 1.173 E-3	0.000 0.000 0.001	2.187 E-13 3.487 E-10 0.002	2.187 E-13 3.487 E-10 0.001

4.3 Concluding remarks

a) It is noticed, from Table 1, that the ML averages are very close to the population parameter values as the sample size increases. Also, ERs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the true parameter values as the sample size increases.
b) The lengths of the CIs of the parameters become narrower as the sample size increases. While in Table 2, It is evident that the variance of p is the smallest one and converges to zero. Also, it is seen from Table 2 that the covariance between k and p is the smallest one.

Table 5: The ML averages of CSALT under Type-II Censoring ($\boldsymbol{n_1 = 125, n_2 = 125}$, $\boldsymbol{r_1}$=30, $\boldsymbol{r_2}$=30, $\boldsymbol{V_1} = 1, \boldsymbol{V_2} = 1.5, \boldsymbol{c = 0.05, p = 0.3\ and\ k = 0.031}$) using Real Data

Parameters	**C**	**K**	**P**	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$
averages	5 E-6	0.031	0.3	4.71 E-6	5.31 E-6

Table 6: Estimates of $\boldsymbol{\alpha_u}$ and $R_u(t_0)$ under usual Conditions $\boldsymbol{V_1 = .5}$ and $\boldsymbol{V_2 = .5}$ for different time $\boldsymbol{t_0}$ at different sample sizes based on Type II censoring using Real Data

c	K	P	$\hat{\alpha}_u$	t_0	$\hat{R}_u(t_0)$
0.05	0.031	0.3	0.038	.01	1.263E-6
				.06	7.285E-7
				.11	5.813E-7
				.16	4.615E-7

Table 7: ML predictive of the future observation based on Type II censoring under two-sample prediction ($\boldsymbol{n_1 = 125, n_2 = 125}$, $\boldsymbol{r_1}$=30, $\boldsymbol{r_2}$=30, $\boldsymbol{V_1} = 1, \boldsymbol{V_2} = 1.5, \boldsymbol{c = 0.05, p = 0.3\ and\ k = 0.031}$) using Real Data

s	$\hat{t}_{(s)}(ML)$
1	3.271E-15
13	4.624E-12
25	6.806E-6

c) From Table3and 6 the estimated values of the rf decreases when the time t_0 increases. The results get better when the sample size increases.

d) The length of the first future order statistic is smaller than the length of the last future order statistic. [Tables 4 and 7].

e) The ML intervals include the predictive values (between the lower limit (LL) and upper limit (UL)).

5. General conclusion

For products having a high reliability, the test of product life under usual conditions often requires a long period of time. So, ALT is used to facilitate estimating the reliability of the unit in a short period of time. In ALT test items are run only at accelerated conditions, this paper deals with the constant stress ALT in the case of Type II censoring. It is assumed that the lifetime of test units has the MOEBX distribution. ML estimators of the parameters are derived. Point and bounds prediction using the ML prediction method for a future observation based on two-sample prediction are studied. From the results, one can clearly see that the estimates have smaller ER. As the sample size increases the ER and the interval of the parameters decrease. Also, the estimated values of the rf decrease when the time t_0 increases. This indicates that the ML estimates provide asymptotically normally distributed and consistent estimators for the parameters.

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التقدير والتنبؤ لإختبارات الحياة المعجلة بإستخدام الضغط الثابت لمارشل أولكن الممتد للنوع العاشر لتوزيع بيير للمراقبه من النوع الثانى

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الملخص العربي

فى هذا البحث تم تقدير معالم التوزيع باستخدام طريقة الإمكان الأكبر (maximum likelihood method) عندما تكون العينات مراقبة من النوع الثانى (Type II censored samples) وهذه الإختبارات المعجلة تتم عند ضغط ثابت (constant stress accelerated life testing) فى هذا البحث حيث تكون فيه العينات مستقلة، وقد وجد أن مقدرات الإمكان الأكبر لا تكون فى صورة صريحة ولذلك تم استخدام إسلوب المحاكاة مونت كارلو(Monte Carlo simulation) باستخدام (Mathcad 15) لحساب مقدرات الإمكان الأكبروقد أُجريت المعالجة الرقمية على ثلاث مجتمعات من التوزيع محل الدراسة باختيار عينات مختلفة وقيم مختلفة للمعالم وعند نسبة مراقبة مختلفه من حجم العينة وقد تم استخدام المحاكاة لحساب تقديرات متوسط مربعات الخطأ mean square error للمقدرات الناتجة وباستخدام خاصية الثبات التى تتميز بها مقدرات الإمكان الأكبر تم تقدير دالة الصلاحية فى حالة الضغط الثابت المعجل وأيضاً تم إيجاد مقدار التحيز المطلق لكلٍ منهما وأيضاً تم استنتاج التقدير بفترة (confidence interval) وطول هذه الفترة (Length of interval) لكل معالم التوزيع. كما تم الحصول على تنبؤلمشاهدة جديدة بإستخدام إسلوب العينتتين two sample prediction وذلك باستخدام طريقة الإمكان الأكبرمن خلال التقدير (بنقطة وفتره). وتم أيضاً إستنتاج حدى فترة التنبؤ للقراءة الصغرى المستقبلية minimum future observable وحدى فترة التنبؤ للقراءة العظمى المستقبلية maximum future observable وحدى فترة التنبؤ للقراءة الوسطى المستقبلية فى الحالة الفردية median future observable كحالات خاصة. ايضا تم التطبيق باستخدم بيانات فعلية على هذا التوزيع وقد لوحظ ان النتائج متشابهه مع النتائج النظرية.