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H. Serag
Dept of Mathematics , Faculty of Science(Men), Al-Azhar University , Cairo

A. Al saban
Department of Mathematics, Faculty of Science, Helwan University, Helwan City, Cairo, Egypt,
ahmedalsaban@yahoo.com

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DISTRIBUTED CONTROL FOR COOPERATIVE ELLIPTIC SYSTEMS UNDER CONJUGATION CONDITIONS

H. M. Serag[1] and A. A. Alsaban[2]

[1]*Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (1884), Cairo, Egypt.*

E- Mail: serraghm@yahoo.com.

[2]*Department of Mathematics, Faculty of Science, Helwan University, Helwan City, Cairo, Egypt.*

E- Mail: ahmedalsaban@yahoo.com.

ABSTRACT

The cooperative elliptic systems involving Laplace operator defined on bounded, continuous and strictly Lipschitz domains of $\mathbb{R}^n$ *and* under conjugation conditions are considered. First, the existence and uniqueness of the state for these systems with Dirichlet and conjugation conditions is proved, then the set of equations and inequalities that characterizes the distributed control of these systems is found. The problem with Neumann conditions is also discussed.

Key words: Cooperative elliptic systems - Conjugation conditions - Dirichlet and Neumann conditions– Existence and uniqueness of solutions-Distributed control**.**

1. INTRODUCTION

The optimal control of systems governed by finite order partial differential (elliptic, parabolic, and hyperbolic) operators defined on finite dimensional spaces have been studied by Lions [11]. The control problems described by either infinite order operators or operators with an infinite number of variables have been discussed by Gali et al [3-8]. These results have been extended in [1, 2, 10, 13, 18] to $n \times n$ cooperative and non-cooperative systems. In [19-21], Sergienko and Deineka introduced control problems of distributed systems with conjugation conditions and quadratic cost functions. Here, we consider 2 × 2 cooperative elliptic systems with conjugation conditions. In section two, we first prove the existence and uniqueness of the state of cooperative Dirichlet system with conjugation conditions. We also find the set of equations and inequalities that characterizes the optimal control of this system. Section three is devoted to the optimal control of distributed type for cooperative Neumann problems.

2- DISTRIBUTED CONTROL OF A SYSTEM DESCRIBED BY THE DIRICHLET PROBLEM

In this section, we study the distributed control for the following 2×2 cooperative Dirichlet elliptic systems:

$$\begin{cases} -\Delta y_1 = ay_1 + by_2 + f_1 & \text{in} \quad \Omega = \Omega_1 \cup \Omega_2 \\ -\Delta y_2 = cy_2 + dy_2 + f_2 & \text{in} \quad \Omega = \Omega_1 \cup \Omega_2 \\ \quad y_1 = y_2 = 0 & \text{on} \quad \Gamma, \end{cases} \tag{2.1}$$

under conjugation conditions:

$$\begin{cases} [y_1(u)] = [y_2(u)] = 0 & on \quad \gamma \\ \left[\frac{\partial y_1}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial y_1}{\partial x_j} cos(v, x_i)\right] = 0 & on \quad \gamma \\ \left[\frac{\partial y_2}{\partial v_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial y_2}{\partial x_j} cos(v, x_i)\right] = 0 & on \quad \gamma \\ \left\{\frac{\partial y_1}{\partial v_A}\right\}^{\pm} = \left\{\sum_{i,j=1}^{n} \frac{\partial y_1}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r[y_1] & on \quad \gamma \\ \left\{\frac{\partial y_2}{\partial v_A}\right\}^{\pm} = \left\{\sum_{i,j=1}^{n} \frac{\partial y_2}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r[y_2] & on \quad \gamma, \end{cases} \tag{2.2}$$

where Ω_1 and Ω_2, with boundary $\partial\Omega_1$ and $\partial\Omega_2$ respectively, are bounded, continuous and strictly Lipchitz domains from n- dimensional Euclidean space $\mathbb{R}^n$ such that:

$$\Omega = \Omega_1 \cup \Omega_2, \ \Omega_1 \cap \Omega_2 = \emptyset,$$

$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)/\gamma$, is boundary of Ω, $\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset, \gamma = \gamma^+ \cup \gamma^-$,

$$\partial\Omega_1 \cap \gamma = \gamma^+, \ \partial\Omega_2 \cap \gamma = \gamma^-, \ f_1, f_2 \in L^2(\Omega),$$

$0 \le r = r(x) \le r_1$, $r \in C(\gamma)$, r_1 is a positive constant, (2.3)

$\vec{n}$ is an ort of an outer normal to Γ, $[\emptyset] = \emptyset^+ - \emptyset^-$,

$$\emptyset^+ = \{\emptyset\}^+ = \emptyset(x) \quad \text{for} \quad x \in \gamma^+,$$

$$\emptyset^- = \{\emptyset\}^- = \emptyset(x) \quad \text{for} \quad x \in \gamma^-,$$

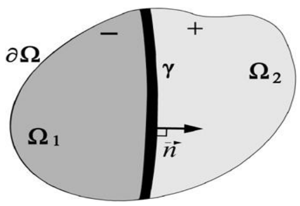

Figure 1: 1

and

a, b, c and d are given numbers such that b, c > 0. (2.4)

Such systems where (2.4) is satisfied is called cooperative systems. They appear in some biological and physical problems [2].

We first prove the existence of the state of the system (2.1)-(2.2) under the following conditions:

$$\begin{cases} a < \mu, d < \mu \\ (\mu - a)(\mu - d) > bc, \end{cases} \tag{2.5}$$

where μ is a positive constant determined by Friedrich inequality:

$$\mu \int_{\Omega} |y|^2\, dx \leq \int_{\Omega} |\nabla y|^2\, dx \tag{2.6}$$

Then, we prove the existence of distributed control for this system; and we find the set of equations and inequalities that characterizes this distributed control.

Existence and uniqueness of the state

By cartesian product, we have the following chain of Sobolev spaces :

$$(H_0^1(\Omega))^2 \subseteq (L^2(\Omega))^2 \subseteq (H^{-1}(\Omega))^2$$

On $(H_0^1(\Omega))^2$, we define the bilinear form:

$$a(y,\psi) = \int_{\Omega} \nabla y_1 \nabla \psi_1\, dx + \int_{\Omega} \nabla y_2 \nabla \psi_2\, dx - \int_{\Omega} (\, a y_1 \psi_1 + b y_2 \psi_1 + c y_1 \psi_2 + d y_2 \psi_2\,) dx$$

$$+ \int_{\gamma} r\,[y_1][\psi_1] d\gamma + \int_{\gamma} r\,[y_2][\psi_2] d\gamma \tag{2.7}$$

Lemma 2.1 The bilinear form (2.7) is coercive on $(H_0^1(\Omega))^2$; that is , there exists a positive constant C such that

$$a(y,y) \geq C \|y\|^2_{(H_0^1(\Omega))^2} \quad \forall y = \{y_1 ,\ y_2\} \in (H_0^1(\Omega))^2 \tag{2.8}$$

Proof

Choose m is large enough such that $a + m > 0$ and $d + m > 0$.

Then,

$$a(y,y) = \frac{1}{b}\int_{\Omega} (|\nabla y_1|^2 + m|y_1|^2)\, dx + \frac{1}{c}\int_{\Omega} (|\nabla y_2|^2 + m|y_2|^2) dx - \frac{a+m}{b}\int_{\Omega} |y_1|^2\, dx$$

$$-\frac{d+m}{c}\int_{\Omega} |y_2|^2\, dx - 2\int_{\Omega} y_1 y_2 dx + \int_{\gamma} r\,[y_1]^2 d\gamma + \int_{\gamma} r\,[y_2]^2\, d\gamma.$$

From (2.3), we get

$$a(y,y) \geq \frac{1}{b}\int_{\Omega} (|\nabla y_1|^2 + m|y_1|^2)\, dx + \frac{1}{c}\int_{\Omega} (|\nabla y_2|^2 + m|y_2|^2) dx - \frac{a+m}{b}\int_{\Omega} |y_1|^2\, dx$$

$$-\frac{d+m}{c}\int_{\Omega} |y_2|^2\, dx - 2\int_{\Omega} y_1 y_2 dx\ .$$

By Cauchy Schwartz inequality

$$a(y,y) \geq \frac{1}{b}\int_{\Omega} (|\nabla y_1|^2 + m|y_1|^2)\, dx + \frac{1}{c}\int_{\Omega} (|\nabla y_2|^2 + m|y_2|^2) dx - \frac{a+m}{b}\int_{\Omega} |y_1|^2\, dx$$

$$-\frac{d+m}{c}\int_{\Omega}|y_2|^2\,dx-2\left(\int_{\Omega}|y_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega}|y_2|^2\,dx\right)^{\frac{1}{2}}$$

From (2.6), we deduce

$$a(y,y)\geq\frac{1}{b}\left(1-\frac{a+m}{\mu+m}\right)\|y_1\|^2+\frac{1}{c}\left(1-\frac{d+m}{\mu+m}\right)\|y_2\|^2-\frac{2}{\mu+m}\|y_1\|\|y_2\|$$

Then

$$a(y,y)\geq\frac{1}{b}\left(1-\frac{a+m-b}{\mu+m}\right)\|y_1\|^2+\frac{1}{c}\left(1-\frac{d+m-c}{\mu+m}\right)\|y_2\|^2$$

Therefore (2.5) implies

$$a(y,y)\geq C\left(\|y_1\|^2+\|y_2\|^2\right)$$

$$\geq C\|y\|^2_{(H_0^1(\Omega))^2}\quad\forall y\in(H_0^1(\Omega))^2.$$

Now, let

$$L(\psi)=\int_{\Omega}(f_1\psi_1+f_2\psi_2)\,dx$$

be a continuous linear form on $(H_0^1(\Omega))^2$,Then using Lax Milgram lemma, there exists a unique solution $y\in H_0^1(\Omega))^2$ such that:

$$a(y,\psi)=L(\psi)\quad\forall\psi\in H_0^1(\Omega))^2$$

Then, we have proved the following theorem

Theorem 2.1**.** For f_1 , $f_2\in(L^2(\Omega))^2$ there exists a unique solution $\{y_1,y_2\}\in(H_0^1(\Omega))^2$ for cooperative Dirichlet system (2.1) with conjugation conditions (2.2) if conditions (2.5) are satisfied.

Formulation of the control problem

The space U = $(L^2(\Omega))^2$ is the space of controls.

For a control u = $(u_1,u_2)\in(L^2(\Omega))^2$, the state $y(u)=\{y_1(u),y_2(u)\}$ of the system is given by the solution of

$$\begin{cases}-\Delta y_1(u)=ay_1(u)+by_2(u)+f_1+u_1 & \text{in }\Omega\\ -\Delta y_2(u)=cy_1(u)+dy_2(u)+f_2+u_2 & \text{in }\Omega\\ y_1(u)=y_2(u)=0 & \text{on }\Gamma,\end{cases}\qquad(2.9)$$

under conjugation conditions:

$$\begin{cases}[y_1(u)]=[y_2(u)]=0 & on\ \gamma\\ \left[\sum_{i,j}^{n}\frac{\partial y_1(u)}{\partial x_j}cos(v,x_i)\right]=\left[\sum_{i,j}^{n}\frac{\partial y_2(u)}{\partial x_j}cos(v,x_i)\right]=0 & on\ \gamma\\ \left\{\frac{\partial y_1(u)}{\partial v_A}\right\}^{\pm}=\left\{\sum_{i,j}^{n}\frac{\partial y_1(u)}{\partial x_j}cos(v,x_i)\right\}^{\pm}=r[y_1(u)] & on\ \gamma\\ \left\{\frac{\partial y_2(u)}{\partial v_A}\right\}^{\pm}=\left\{\sum_{i,j}^{n}\frac{\partial y_2(u)}{\partial x_j}cos(v,x_i)\right\}^{\pm}=r[y_2(u)] & on\ \gamma\end{cases}\qquad(2.10)$$

The observation equation is given by:

$$Z(u) = \{Z_1(u), Z_2(u)\} = C\, y(u) = C\,\{y_1(u), y_2(u)\} = \{y_1(u), y_2(u)\}$$

For a given $z_d = \{z_{1d}, z_{2d}\} \in (L^2(\Omega))^2$, the cost function is given by

$$J(v) = \|y_1(v) - z_{1d}\|_{(L^2(\Omega))^2} + \|y_2(v) - z_{2d}\|_{(L^2(\Omega))^2} + (Nv, v)_{(L^2(\Omega))^2}, \tag{2.11}$$

where N is a hermitian positive definite operator such that :

$$(Nv, v)_{(L^2(\Omega))^2} \geq M\|v\|^2_{(L^2(\Omega))^2}, \quad M > 0 \tag{2.12}$$

The control problem then is to:

$$\begin{cases} \text{Find } u = (u_1, u_2) \in u_{ad} \text{ (closed convex subset of } (L^2(\Omega))^2 \text{ such that:} \\ \qquad J(u) = \inf J(v) \qquad \forall\, v \in u_{ad} \end{cases} \tag{2.13}$$

The cost function (2.11) can be written as

$$J(v) = \|y_1(v) - y_1(0) + y_1(0) - z_{1d}\|_{(L^2(\Omega))^2}$$
$$+\|y_2(v) - y_2(0) + y_2(0) - z_{2d}\|_{(L^2(\Omega))^2} + (Nv, v)_{(L^2(\Omega))^2}$$

If we let:

$$\pi(u, v) = \left(y_1(u) - y_1(0)\,, y_1(v) - y_1(0)\right)_{(L^2(\Omega))^2}$$

$$+\left(y_2(u) - y_2(0)\,, y_2(v) - y_2(0)\right)_{(L^2(\Omega))^2} + (Nv, v)_{(L^2(\Omega))^2}$$

and

$$f(v) = \left(z_{1d} - y_1(0)\,, y_1(v) - y_1(0)\right)_{(L^2(\Omega))^2}$$

$$+\left(z_{2d} - y_2(0)\,, y_2(v) - y_2(0)\right)_{(L^2(\Omega))^2}$$

Then

$$J(v) = \pi(v, v) - 2f(v) + \|z_{1d} - y_1(0)\|^2_{(L^2(\Omega))^2} + \|z_{2d} - y_2(0)\|^2_{(L^2(\Omega))^2}$$

Since from (2.12) ,$\pi(v, v) \geq M\|v\|^2_{(H^1(\Omega))^2}$,

then, using the theory of Lions [11], there exists a unique optimal control of problem(2.13); moreover it is characterized by

Theorem 2.2 Let us suppose that (2.8) holds and the cost function is given by (2.11), then the distributed control u is characterized by:

$$\begin{cases} -\Delta p_1 - ap_1 - cp_2 = y_1(u) - z_{1d} & \text{in } \Omega \\ -\Delta p_2 - bp_1 - dp_2 = y_2(u) - z_{2d} & \text{in } \Omega \\ p1 = p2 = 0 & \text{on } \Gamma \\ [p1] = [p2] = 0 & \text{on } \gamma \\ \left[\frac{\partial y_1(u)}{\partial x_j} cos(v, x_i)\right] = \left[\frac{\partial y_2(u)}{\partial x_j} cos(v, x_i)\right] = 0 & on \quad \gamma \\ \left\{\frac{\partial y_1(u)}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r\,[p_1] & on \quad \gamma \\ \left\{\frac{\partial y_2(u)}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r\,[p_2] & on \quad \gamma \\ (p_1, v_1 - u_1) + (p_2, v_2 - u_2) + (Nu, v - u)_{(L^2(\Omega))^2} \geq 0, & \end{cases}$$

together with (2.9) and (2.10) ,where p(u) = {p_1 (u), p_2 (u)} is the adjoint state.

Proof

The optimal control u is characterized by [see 11, 13]:

$$\begin{gathered} \pi(u, v-u) - L(v-u) \geq 0 \qquad \forall\, v \in u_{ad} \\ \left(y_1(u) - y_1(0)\,, y_1(v-u) - y_1(0)\right)_{(L^2(\Omega))^2} \\ +\left(y_2(u) - y_2(0)\,, y_2(v-u) - y_2(0)\right)_{(L^2(\Omega))^2} \\ +(Nu, v-u)_{(L^2(\Gamma))^2} - \left(z_{1d} - y_1(0), y_1(v-u) - y_1(0)\right) \\ -\left(z_{2d} - y_2(0), y_2(v-u) - y_2(0)\right) \geq 0 \end{gathered}$$

Then

$$\begin{gathered} \left(y_1(u) - z_{1d}\,, y_1(v-u) - y_1(0)\right)_{(L^2(\Omega))^2} \\ + \left(y_2(u) - z_{2d}\,, y_2(v-u) - y_2(0)\right)_{(L^2(\Omega))^2} \\ +(Nu, v-u)_{(L^2(\Gamma))^2} \geq 0 \end{gathered} \tag{2.14}$$

Since the model A of the system is given by

$$A\,y(x) = A\,(y_1,y_2)$$

$$= (-\Delta y_1 - ay_1 - by_2,\ -\Delta y_2 - cy_1 - dy_2)$$

and since

$$(A^*p, y) = (p, A\,y),$$

then

$$(p, A\,y) = \left(p_1, -\Delta y_1(u) - ay_1(u) - by_2(u)\right) + \left(p_2, -\Delta y_2(u) - cy_2(u) - dy_2(u)\right)$$

$$= \left(-\Delta p_1(u) - ap_1(u) - cp_2(u)\,, y_1\right) + \left(-\Delta p_2(u) - bp_2(u) - dp_2(u)\,, y_2\right)$$

and since the adjoint state is defined by :

$$\begin{cases} -\Delta p_1 - ap_1 - cp_2 = y_1(u) - z_{1d} & \text{in } \Omega \\ -\Delta p_2 - bp_1 - dp_2 = y_2(u) - z_{2d} & \text{in } \Omega \\ \dfrac{\partial p_1}{\partial v_{A^*}} = \dfrac{\partial p_2}{\partial v_{A^*}} = 0 & on\,\Gamma \\ \left[\dfrac{\partial p_1}{\partial v_{A^*}}\right] = \left[\dfrac{\partial p_2}{\partial v_{A^*}}\right] = 0 & on\,\gamma \\ \left\{\dfrac{\partial p_1}{\partial v_{A^*}}\right\}^{\pm} = r\,[p_1]\,, \left\{\dfrac{\partial p_2}{\partial v_{A^*}}\right\}^{\pm} = r\,[p_2] & on\,\gamma \end{cases}$$

Then (2.14) implies

$$(-\Delta p_1(u) - ap_1(u) - cp_2(u)\,, y_1(v-u) - y_1(0))_{(L^2(\Omega))^2} + \left(-\Delta p_2(u) - bp_1(u) - dp_2(u)\,, y_2(v-u) - y_2(0)\right)_{(L^2(\Omega))^2} + (Nu, v-u)_{(L^2(\Omega))^2} \geq 0$$

Therefore

$$\left(A^* p_1\,, y_1(v-u) - y_1(0)\right)_{(L^2(\Omega))^2} + \left(A^* p_2\,, y_2(v-u) - y_2(0)\right)_{(L^2(\Omega))^2} + (Nu, v-u)_{(L^2(\Omega))^2} \geq 0\,,$$

$$\left(p_1\,, Ay_1(v-u) - Ay_1(0)\right)_{(L^2(\Omega))^2} + \left(p_2\,, Ay_2(v-u) - Ay_2(0)\right)_{(L^2(\Omega))^2} + (Nu, v-u)_{(L^2(\Omega))^2} \geq 0\,.$$

From (2.9), we obtain

$$(p_1, v_1 - u_1)_{(L^2(\Omega))^2} + (p_2, v_2 - u_2)_{(L^2(\Omega))^2} + (Nu, v-u)_{(L^2(\Omega))^2} \geq 0 \quad .$$

Hence

$$\int_\Omega (p_1 + N\,u_1)\,(v_1 - u_1)dx + \int_\Omega (p_2 + N\,u_2)\,(v_2 - u_2)dx \;\geq 0 \quad .$$

3. COOPERATIVE NEUMANN ELLIPTIC SYSTEMS WITH CONJUGATION CONDITIONS

In this section, we consider the following Neumann cooperative elliptic system

$$\begin{cases} -\Delta y_1 = ay_1 + by_2 + f_1 & \text{in } \Omega \quad \subset R^n \\ -\Delta y_2 = cy_1 + dy_2 + f_2 & \text{in } \Omega \quad \subset R^n \\ \dfrac{\partial y_1}{\partial v_A} = g_1\,, \dfrac{\partial y_2}{\partial v_A} = g_2\,, & \text{on} \quad \Gamma \end{cases}$$

with conjugation conditions (2.2) , where $g_i \in (L^2(\Omega))^2$, i=1,2.
We introduce again the bilinear form (2.7) which is coercive on $(H^1(\Omega))^2$, since

$$(H_0^1(\Omega))^2 \subseteq (H^1(\Omega))^2$$

Then by Lax Milgram lemma, for a control $u \in (L^2(\Omega))^2$ the state $(y_1(u), y_2(u)) \in (H^1(\Omega))^2$ of the system is given by the solution of

$$a(y(u), \psi(u)) = L_N(\psi) \quad \forall \psi \in (H^1(\Omega))^2$$

Where

$$L_N(\psi) = \int_\Omega \{(f_1 + u_1)\,\psi_1 + (f_2 + u_2)\,\psi_2\}dx + \int_\Gamma (g_1\psi_1 + g_2\psi_2)d\Gamma$$

is a continuous linear form defined on $(H^1(\Omega))^2$

then, applying green formula, we get

$$\int_\Omega (-\Delta y_1)\psi_1 dx + \int_\Omega (-\Delta y_2)\psi_2 dx + \int_\Omega (ay_1\psi_1 + by_2\psi_1 + cy_1\psi_2 + dy_2\psi_2)dx$$
$$-\int_\Gamma \frac{\partial y_1}{\partial v_A}\psi_1 d\Gamma - \int_\Gamma \frac{\partial y_2}{\partial v_A}\psi_2 d\Gamma - \int_\gamma \frac{\partial y_1}{\partial v_A}\psi_1 d\gamma$$
$$-\int_\gamma \frac{\partial y_2}{\partial v_A}\psi_2 d\gamma + \int_\gamma r[y_1][\psi_1]d\gamma + \int_\gamma r[y_2][\psi_2]d\gamma$$
$$= \int_\Omega \{(f_1 + u_1)\,\psi_1 + (f_2 + u_2)\,\psi_2\}dx + \int_\Gamma (g_1\psi_1 + g_2\psi_2)d\Gamma$$

$$\int_\Omega (-\Delta y_1 - ay_1 - by_2)\psi_1 dx + \int_\Omega (-\Delta y_2 - cy_1 - dy_2)\psi_2 dx$$
$$-\int_\Gamma \frac{\partial y_1}{\partial v_A}\psi_1 d\gamma - \int_\Gamma \frac{\partial y_2}{\partial v_A}\psi_2 d\gamma - \int_\gamma \frac{\partial y_1}{\partial v_A}\psi_1 d\gamma$$
$$-\int_\gamma \frac{\partial y_2}{\partial v_A}\psi_2 d\gamma + \int_\gamma r[y_1][\psi_1]d\gamma + \int_\gamma r[y_2][\psi_2]d\gamma$$
$$= \int_\Omega \{(f_1 + u_1)\,\psi_1 + (f_2 + u_2)\,\psi_2\}dx + \int_\Gamma (g_1\psi_1 + g_2\psi_2)d\Gamma$$

i.e the state $(y_1(u), y_2(u)) \in (H^1(\Omega))^2$ is given by

$$\begin{cases} -\Delta y_1(u) = ay_1(u) + by_2(u) + f_1 + u_1 & \text{in } \Omega \subset R^n \\ -\Delta y_2(u) = cy_1(u) + dy_2(u) + f_2 + u_2 & \text{in } \Omega \subset R^n \\ \dfrac{\partial y_1}{\partial v_A} = g_1 \quad , \dfrac{\partial y_2}{\partial v_A} = g_2 \,, & \text{on } \Gamma \end{cases}$$

under conjugation conditions:

$$\begin{cases} \left[\dfrac{\partial y_1(u)}{\partial v_A}\right] = \left[\displaystyle\sum_{i,j}^{n} \dfrac{\partial y_1(u)}{\partial x_j} cos(v, x_i)\right] = 0 & on \quad \gamma \\ \left[\dfrac{\partial y_2(u)}{\partial v_A}\right] = \left[\displaystyle\sum_{i,j}^{n} \dfrac{\partial y_2(u)}{\partial x_j} cos(v, x_i)\right] = 0 & on \quad \gamma \\ \left\{\dfrac{\partial y_1(u)}{\partial v_A}\right\}^{\pm} = \left\{\displaystyle\sum_{i,j}^{n} \dfrac{\partial y_1(u)}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r[y_1(u)] & on \quad \gamma \\ \left\{\dfrac{\partial y_2(u)}{\partial v_A}\right\}^{\pm} = \left\{\displaystyle\sum_{i,j}^{n} \dfrac{\partial y_2(u)}{\partial x_j} cos(v, x_i)\right\}^{\pm} = r[y_2(u)] & on \quad \gamma \end{cases} \tag{3.3}$$

For a given $z_d = \{z_{1d}, z_{2d}\} \in (L^2(\Omega))^2$, The cost function is given by (2.11)
As in theorem(2.3), we can prove

Theorem 3.1 If the cost function is given by (2.11), there exists a unique optimal control u= $(u_1, u_2) \in (L^2(\Omega))^2$, such that:

$$J(u) \le J(v) \quad \forall v \in u_{ad} \subset (L^2(\Omega))^2;$$

Moreover it is characterized by the following equations and inequalities

$$\begin{cases} -\Delta p_1 - a p_1 - c p_2 = y_1(u) - z_{1d} & in \ \Omega \\ -\Delta p_2 - b p_1 - d p_2 = y_2(u) - z_{2d} & in \ \Omega \\ \dfrac{\partial p_1}{\partial v_{A^*}} = \dfrac{\partial p_2}{\partial v_{A^*}} = 0 & on \, \Gamma \\ \left[\dfrac{\partial p_1}{\partial v_{A^*}}\right] = \left[\dfrac{\partial p_2}{\partial v_{A^*}}\right] = 0 & on \, \gamma \\ \left\{\dfrac{\partial p_1}{\partial v_{A^*}}\right\}^{\pm} = r\,[p_1], \left\{\dfrac{\partial p_2}{\partial v_{A^*}}\right\}^{\pm} = r\,[p_2] & on \, \gamma \end{cases}$$

$$\int_{\Omega} (p_1 + N u_1)(v_1 - u_1)dx + \int_{\Omega} (p_2 + N u_2)(v_2 - u_2)dx \ \ge 0$$

together with (3.2) and (3.3)

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